Comparison of Models for Equivariant Operads

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Contents

Abstract

We introduce and explore multiple categorical frameworks which allow for a general investigation of the homotopy theory of equivariant operads. We extend the Cisinski-Moerdijk-Weiss theory of dendroidal sets by suitably generalizing the category of trees in order to record equivariant composition information. This formalism of G-trees is then applied to a rebuilt free operad monad in order to endow the category of G-operads in $\mathcal{V}$ with an $\mathcal{F}$-semi-model structure for any weak indexing system $\mathcal{F}$ and fairly general model categories $\mathcal{V}$; as a consequence, we prove that all indexing systems can be realized as N_∞-operads, confirming a conjecture of Blumberg-Hill.

Also using G-trees, we define appropriate notions of inner G-horns and G-∞-operads. Inspired by the internal algebra of G-trees and G-∞-operads, we extend G-operads to the new algebraic notion of genuine equivariant operads, which allow us to record the equivariance of our operadic compositions, while removing rigidity conditions on fixed points without relaxing the strictness of the composition laws. Lastly show that there is a natural homotopy strictification functor, sending G-∞-operads to an associated genuine G-operad.

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Introduction

Operads and related algebraic objects can encode a wide variety of structures - including (commutative) monoids, Lie algebras, and modules over an algebra - by providing lists of "n-ary operations" for each n. . As such, they have proved useful in characterizing properties, analyzing theories, and ultimately answering questions in homotopy theory (e.g. [May72, HHR16, EKMM97]).

Classically, in topological spaces, the E_∞-operads of May [May72] provide a description of "topological homotopy commutative monoids", where algebras have (homotopy unique) maps $R^n \to R$ which are associative and unital up to all higher homotopies; moreover, (group-like) E_∞-spaces have the geometric and stable structure of an infinite loop space. As such, these are important objects in (stable) homotopy theory. They are characterized by the property that each space $\mathcal{O}(n)$ is Σ_n-free and contractible.

Equivariantly, the story becomes more nuanced. Given a finite group G, commutative G-ring spectra are additionally endowed with "norm maps" $N^A R \to R$ for any H-set A, where H is a subgroup of G and $N^A R$ is a "multiplicative induction" of R, with the diagonal group action being twisted by the action on A. These maps were of paramount importance to the solution in [HHR16] to the Kervaire Invariant One problem, and continue to be important in many areas of equivariant homotopy theory. With this in mind, any model of "equivariant E_∞-operads" chosen should encode all of these norm maps.

A natural choice would be to make the following definition:

*A G-operad $\mathcal{O}$ is a G-E_∞-*operad *if each space $\mathcal{O}(n)$ is Σ_n-free and G-contractible.* (✩)

However, the naïve first example, created by endowing an E_∞-operad with the trivial G-action, does *not* encode norm maps for any H-sets except the trivial ones. Analogously, in G-spaces, it only encodes "infinite loop spaces with G-action", as opposed to the significantly stronger notion of "equivariant infinite loop spaces" (which have deloopings for all *representation spheres*).

The correct characterization was identified by Constenoble-Waner [CW91]. The main issue is that ($\star$) does *not* determine a unique $G \times \Sigma_n$-homotopy type. In particular, it only says that $\mathcal{O}(n)^\Gamma = \varnothing$ if $\Gamma \cap \Sigma_n \neq \{e\}$, and $\mathcal{O}(n)^\Gamma \simeq *$ for $\Gamma \leq G$. This does not specify what occurs when Γ is a non-trivial *graph subgroup*, the graph of a partial homomorphism $G \hookleftarrow H \to \Sigma_n$. Dual to the naive choice above, [CW91] define a *genuine G-E_∞-operad* to have $\mathcal{O}(n)$ contractible for *all* of these graph subgroups Γ. Indeed, algebras over these have the full rich structure we desire: G-E_∞-spectra have all norm maps, and (group-like) G-E_∞-spaces model equivariant infinite loop spaces.

Moreover, Blumberg-Hill show in [BH15] that there exists an entire *lattice* of operads scaling between the naive and genuine version; they call these *N_∞-operads* ("N" for "norm"). In terms of graph subgroups, they show that each N_∞-operad $\mathcal{O}$ has $\mathcal{O}(n)^\Gamma \simeq *$ for a specified families $\mathcal{F}_n$ of graph subgroups of $G \times \Sigma_n$. Moreover, the associated collection of families $\mathcal{F} = \{\mathcal{F}_n\}$ satisfies a variety of properties, which they use to define an *indexing system*. In fact, they show that the homotopy category $\mathrm{Ho}(N_\infty\text{-}\mathsf{Op})$ maps fully-faithfully into the poset $\mathbb{I}$ of indexing systems, and conjecture that this map is additionally essentially surjective.

The upshot of this discussion is that the homotopy theory of (topological) G-operads must *distinguish* each of these N_∞-operads from each other, and hence must probe all of the fixed points for all graph subgroups of each $G \times \Sigma_n$.

Now, there is already machinery in place (e.g. [Ste16]) which lifts a model structure on a category $\mathcal{C}$ to one on $\mathcal{C}^G$. However, this will be insufficient, as it will only see the fixed points $\mathcal{O}(n)^H$, of the trivial graph subgroups. Similarly, the axiomatic approach of Berger-Moerdijk [BM03, BM07, BM13] is also insufficient, as they build a model structure on operads by lifting a projective model structure on symmetric sequences, which has the affect of ignoring any of the interesting fixed-point data.

In this thesis, we will provide the foundations to solve this problem and to better analyze the homotopy theory of equivariant operads.

First, we expand and generalize the presentation of (G-)operads themselves, by rebuilding the free operad monad as a particular left Kan extension:

Theorem (3.2.7; see also 4.2.40). *The free operad monad $\mathbb{F}$ on a symmetric sequence X is the left Kan extension of the nerve-evaluation map for X over the valence functor of trees.*

This allows for a new description of a filtration of *cellular extensions* $\mathcal{P} \to \mathcal{P}[u]$ of $\mathcal{V}$-operads, for $\mathcal{V}$ a fairly general category (Theorem 3.5.10). These cellular extensions are particular pushouts which have been extensively studied, as the Transfer Principle of Kan or Schwede-Shipley says that these are of paramount importance to lifting model structures across free-forgetful adjunctions $F : \mathsf{Alg}_F(\mathcal{C}) \leftrightarrows \mathcal{C} : F$. These pushouts, and filtrations of them, have been studied in various forms (e.g. [Hir03, SS00, Spi01, Whi14a, Whi14b, WY15, Har09, Har10, Per16, HP15], to name just a few).

The Transfer Principle says, in particular, that if each step $\mathcal{P}_i \to \mathcal{P}_{i+1}$ in the filtration is a (trivial) cofibration for any F-algebra $\mathcal{P}$, then the model structure on $\mathcal{C}$ lifts to the category $\mathsf{Alg}_F(\mathcal{C})$ of F-algebras. If instead, this property only holds for *cofibrant* F-algebras $\mathcal{P}$, then the model structure on $\mathcal{C}$ lifts to a *semi*-model structure, a weaker notion.

Using our filtration, we show the following.

Theorem (4.3.9). *Let $\mathcal{V}$ satisfy* Assumption 1*, and let $\mathcal{F}$ be a weak indexing system. Then the category $\mathcal{V}\mathsf{Op}^G$ of single-coloured G-operads in $\mathcal{V}$ can be endowed with the $\mathcal{F}$-semi-model structure, where $f : \mathcal{O} \to \mathcal{P}$ is a weak equivalence or fibration if, for any n, $f(n)^\Gamma$ is so for any and all $\Gamma \in \mathcal{F}_n$.*

As an immediate corollary, we have

Corollary (4.3.11). *For any weak indexing system $\mathcal{F}$, there exists an operad $N^{\mathcal{F}}$ such that*

$N^{\mathcal{F}}(n)^{\Gamma} \simeq *$ if $\Gamma \in \mathcal{F}(n)$, and is empty otherwise. In particular, $\mathrm{Ho}(N_\infty\text{-}\mathsf{Op}) \to \mathbb{I}$ *is an equivalence of categories.*

The notion of "weak indexing system" is a relaxing of the definition in Blumberg-Hill, which naturally falls out of a second lens through which this thesis studies equivariant operads. Generalizing the story of Moerdijk-Weiss and Cisinski-Moerdijk, we define and explore the categories of G-trees and equivariant dendroidal sets. CMW used the dendroidal category of trees Ω to probe the combinatorics of operadic compositions, using grafting of trees and inner face maps (which is itself a generalization of the story of using Δ to probe the combinatorics of categorical compositions). In particular, they found a natural combinatorial model of ∞-operads as objects in the presheaf category $\mathsf{dSet} = \mathsf{Set}^{\Omega^{op}}$ satisfying a lifting condition (for which operads were precisely those which satisfied the lifting condition *strictly*).

Our category Ω_G of G-trees (see 5.1.17) subtly extends and amalgamates $\Omega \times O_G$, where O_G is the category of G-orbits and G-maps, and is used to probe and record the *equivariance* of compositions of operads via grafting and inner face maps. Weak indexing systems will correspond precisely to those subcategories which are closed under inner faces and "pullbacks" (5.1.55).

We use Ω_G to define G-∞-operads in $\mathsf{dSet}^G = \mathsf{dSet}^{(G\times\Omega)^{op}}$ via a lifting condition (for which G-operads are precisely those which satisfy the lifting condition *strictly*; see 5.2.37):

Definition (5.2.18)**.** $X \in \mathsf{dSet}^G$ is called a *G-∞-operad* if it has the right lifting property against all inner horn inclusions $\Lambda^{G\cdot e}[T] \hookrightarrow \Omega[T]$ for all G-trees T and edges $e \in T$.

Further, Pereira [Per16] has shown that, as in the non-equivariant case, the features which define G-∞-operads yield a model structure on dSet^G with fibrant objects precisely the G-∞-operads.

Furthermore, in dSet, we can naturally associate to every ∞-operad a "strict" operad in Op called its *homotopy operad*. While the parallel construction on G-∞-operads exists in dSet^G, the most natural target isn't Op^G, as forcing the functor to land here requires us to forget vast amounts of equivariant structure. Instead, G-∞-operads inspired the definition of a more general algebraic structure than G-operads, which we call *genuine equivariant operads* (6), denoted Op_G. These should be thought of as "coefficient operads", in the sense that we have an adjunction $\mathcal{V}\mathsf{Op}^G \leftrightarrows \mathcal{V}\mathsf{Op}_G$ parallel to $\mathsf{Top}^G \leftrightarrows \mathsf{Top}^{O_G^{op}}$; in particular, we fully expect the above adjunction to be a Quillen equivalence whenever the model structures exist.

The definition of genuine operads is precisely an extension of the free operad monad built earlier in the thesis, by replacing the use of Ω with Ω_G:

Definition (6.3.7)**.** A *genuine G-operad* is an algebra over the monad $\mathbb{F}_G$ which sends X to the left Kan extension below:

$$\begin{tikzcd}
\mathbb{T}^{op}_{G,0} \arrow[r, "N_X"] \arrow[d, "val"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\
\Sigma^{op}_G \arrow[ur, "\mathrm{Lan}_{val} N_X \simeq \mathbb{F}X"'] &
\end{tikzcd}$$

Moreover, we indeed have a natural strictification functor, sending a G-∞-operad to its "homotopy genuine equivariant operad" of sets (Proposition 6.3.41).

Organization

This thesis is organized as follows.

Chapter 1: Preliminaries will provide a basic introduction to the notion and constructions we will be using throughout the thesis.

Chapter 2: Operads and Trees will recall the various definitions of operads, the two definitions of trees, and the category of dendroidal sets.

Chapter 3: Categorical Constructions on Single-Coloured Operads will repackage and realize the free operad monad,various coproducts of operads, and cellular extensions as left Kan extensions over categories of structured trees. We will also construct the filtration of cellular extensions here.

Chapter 4: Equivariant Homotopy Theory and Equivariant Operads will rigorously discuss and analyze much which was found here in the introduction, and will introduce a free monad for operads with a G-set of colours. At the end, we will announce our semi-model structure and N_∞-realization results.

Chapter 5: Equivariant Dendroidal Sets will introduce and investigate the concepts of G-trees and equivariant dendroidal sets, define G-∞-operads, and prove the technical cofibrancy results needed for the main theorems.

Chapter 6: Genuine Equivariant Operads will define and explore various models for genuine equivariant operads, as well as compare them with each other and with normal G-operads.

Appendix A: Kan Extensions collects various technical results about the naturality of the left Kan extension with respect to other categorical constructions.

Appendix B: Counterexamples to the Candidates for $N^{\mathcal{F}}$ shows that an initial guess by Blumberg-Hill and May to realize an N_∞-operad for any indexing system $\mathcal{F}$ fails both to have the correct homotopy type and at being an operad for general groups G.

Chapter 1

Preliminaries

The main object of study in this thesis will be *operads*. An operad is a gadget used to encode "generalized multiplications", and has been incredibly fruitful in describing structure and enlightening theories in algebraic topology, algebraic geometry, homological algebra, and representation theory, among other fields.

We begin by setting the language, notation, and certain constructions in category theory, which will be useful in the rest of the thesis.

1.1 Category Theory

For the duration of this thesis, we will be working with a *closed symmetric monoidal category* $\mathcal{V} = (\mathcal{V}, \otimes, I)$. In this section, we will briefly outline some of the major features and structures of such a category. For a more complete analysis, see [Mac71] or [Hov99].

We begin by establishing the notations of category theory we will be using. Throughout, we will ignore any "smallness" questions, and assume we are working in a sufficiently large set universe. Given a category $\mathcal{V}$, we denote its set of objects by $\mathrm{Ob}(\mathcal{V})$, but in general we will write $V \in \mathcal{V}$ to mean $V \in \mathrm{Ob}(\mathcal{V})$. The set of arrows $V \to W$ in $\mathcal{V}$ will be denoted $\mathcal{V}(V, W)$. We will use $V \simeq W$ to denote that V and W are isomorphic.

We say our category $\mathcal{V}$ is equipped with a *monoidal structure* if we have a "monoidal product" functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and a specified element $I \in \mathcal{V}$, such that $\otimes$ is associative and unital. We additionally call this structure

- *closed* if each functor $V \otimes (-) : \mathcal{V} \to \mathcal{V}$ has a right adjoint; we denote this adjoint $\mathsf{Hom}(V, W)$ the "internal Hom";
- *symmetric* if we have a natural twisting isomorphism $V \otimes W \simeq W \otimes V$ satisfying coherency conditions;
- *(co)Cartesian* if the monoidal product is just the categorical (co)product of $\mathcal{V}$.

Notation 1.1.1. In general, we will write adjoints $R : \mathcal{C} \leftrightarrows \mathcal{D} : L$ with the right adjoint on the left; if this is impossible, the arrow representing the right adjoint will always be below the arrow of the left adjoint.

Example 1.1.2. If $\mathcal{V}$ has all (co)limits, then $\mathcal{V}$ can be equipped with the (co)Cartesian monoidal product.

Remark 1.1.3. If $(\mathcal{V}, \otimes)$ is closed monoidal, then, in particular, $V \otimes (-)$ commutes with all colimits.

Given a second category $\mathcal{D}$, we will denote the category of functors $\mathcal{D} \to \mathcal{V}$ and natural transformations between them by $\mathcal{V}^{\mathcal{D}}$.

1.1.1 Internal Categorical Constructions

Let F denote the category of *finite sets*, and F_0 the wide subcategory of sets and isomorphisms. We define an "action" of F on $\mathcal{V}$ for any category with coproducts.

Definition 1.1.4. Given a set $A \in \mathsf{F}$, let

$$A \cdot (-) : \mathcal{V} \to \mathcal{V}$$

be the functor which sends V to $A \cdot V = \amalg_A V$, called the *copower* of A with V.

Copowers can be viewed as a certain class of colimits. We describe two other important classes.

Definition 1.1.5. Given a functor $F : \mathcal{V}^{op} \times \mathcal{V} \to \mathcal{V}$, define the *coend* to be the coequalizer of the maps

$$\coprod_{V' \to V''} F(V'', V') \rightrightarrows \coprod_{V} F(V, V)$$

and is denoted with an integral as $\displaystyle\int^{V \in \mathcal{V}} F(V, V)$.

A particular type of coend will be very important to our discussion below, namely left Kan extensions.

Definition 1.1.6. Given a span of category $\mathcal{D} \xleftarrow{i} \mathcal{C} \xrightarrow{X} \mathcal{V}$, we define the *left Kan extension* of X over i, denoted $\mathrm{Lan}_i\, X$, to be the universal functor $\mathcal{D} \to \mathcal{V}$ with a natural transformation $\alpha : X \to \mathrm{Lan}_i\, X \circ i$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "X"] \arrow[d, "i"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathcal{V} \\
\mathcal{D} \arrow[ur, "\mathrm{Lan}_i\, X"']
\end{tikzcd}$$

If $\mathcal{V}$ has enough colimits, this can be described "point-wise":

$$`\,\mathrm{Lan}_i\, X(d) \simeq \int^{c \in \mathcal{C}} X(c) \cdot \mathcal{D}(c, d) =: \operatorname*{colim}_{\substack{\mathcal{C} \downarrow d \\ c \to d}} X(c).$$

Equivalently, Lan is a particular left adjoint:

Lemma 1.1.7. *For any $Z : \mathcal{D} \to \mathcal{V}$, we have $\mathcal{V}^{\mathcal{D}}(\mathrm{Lan}_i\, X, Z) = \mathcal{V}^{\mathcal{C}}(X, Zi)$.*

We will discuss some of the naturality of this construction in Appendix A. However, an immediate application is the following result, known either as "Yoneda reduction" or the "co-Yoneda Lemma", among others.

Lemma 1.1.8. *[Yoneda Reduction] Given a functor $F : \mathcal{C} \to \mathcal{D}$ and any $c \in C$, we have*

$$F(c) \simeq \int^{c' \in \mathcal{C}} F(c') \times \mathcal{C}(c', c).$$

1.1.2 External Categorical Constructions

One of the most powerful categorical tools for synthesizing information and building new categories is the Grothendieck construction.

Definition 1.1.9. Given a functor $F : \mathcal{C}^{op} \to \mathsf{Cat}$, define the *Grothendieck construction* $\int_{\mathcal{C}} F$ to be the category with objects all pairs (c, X) with $c \in \mathcal{C}$ and $X \in F(c)$, and arrows $(c, X) \to (c', X')$ pairs of maps (f, φ) with $f : c \to c'$ an arrow in $\mathcal{C}$ and $\varphi : X \to F(f)(X')$ an arrow in $F(c)$.

We have a canonical functor $\int_{\mathcal{C}} F \to \mathcal{C}$ sending (c, X) to c, which is called a *Grothendieck fibration.*

Example 1.1.10. We can think of the Grothendieck construction as a generalization of the wreath product of groups (among other things). In particular, given a group G, viewed categorically as each having a single object with a group of self-morphisms, then the functor $\Sigma_n \to \mathsf{Cat}$ sending the single object to $G^{\times n}$ will have Grothendieck construction $\int_{\Sigma_n} G^{\times n} \simeq \Sigma_n \wr G$.

Lemma 1.1.11. *$\int_{\mathcal{C}}(-)$ is functorial in the category of functors $\mathsf{Fun}(\mathcal{C}^{op}, \mathsf{Cat})$.*

Proof. This follows immediately from unpacking the definitions. Given $\Phi : F \Rightarrow G$, define $\int \Phi$ on objects by $(c, X) \mapsto (c, \Phi(c)(X))$ and on arrows by $(f, \varphi) \mapsto (f, \Phi(c)(f))$; the map on arrows is well-defined by the coherence requirements on Φ. $\square$

Additionally, we will reference "nerve-realization" adjunctions, generalizing the adjunction $N : \mathsf{Cat} \leftrightarrows \mathsf{sSet} : \tau$.

Lemma 1.1.12. *Given any small category $\mathcal{D}$ and cocomplete $\mathcal{V}$, and a functor $\mathcal{D}[-] : \mathcal{D} \to \mathcal{V}$, there exists a unique colimit-preserving functor $\tau : \mathsf{Set}^{\mathcal{D}^{op}} \to \mathcal{V}$ such that $\tau(\mathcal{D}(-, d)) = \mathcal{D}[d]$. Moreover, this forms an adjunction*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "{\mathcal{D}[-]}"] \arrow[d, "Y"'] & \mathcal{V} \arrow[dl, shift left, "N"] \\
\mathsf{Set}^{\mathcal{D}^{op}} \arrow[ur, shift left, "\tau"] &
\end{tikzcd}$$

1.2 Model Categories

We very briefly discuss the main features of a Quillen model category; see [Qui67] or [Hov99] for a more thorough analysis.

In general, a model category structure on $\mathcal{C}$ allows a mathematician to "do homotopy theory" on $\mathcal{C}$. More specifically, it allows for a well-behaved localization of $\mathcal{C}$ at a specific class of "weak equivalences". Importantly, it is defined in terms of "lifting properties":

Definition 1.2.1. The pair (f, g) satisfy the *lifting condition* if for all commuting squares

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r] \arrow[ur, dotted, "\exists"] & \cdot \end{tikzcd}$$

there exists a dotted arrow as denoted making the two triangles commute.

We say f has the *left lifting property* against g, and dually that g has the *right lifting property* against f.

We say this is *strict* if the lift is always unique, and say f has the strict left lifting property against g, and dually.

Definition 1.2.2. A *model structure* on a category $\mathcal{C}$ consists of the data of

- three classes of morphisms called *cofibrations*, *fibrations*, and *weak equivalences*, and
- two functorial factorization systems

such that

M1 $\mathcal{C}$ is complete and cocomplete;

M2 weak equivalences are closed out of "2-out-of-3";

M3 all three subcategories are closed under retracts;

M4 *trivial cofibrations*, defined to be maps which are both cofibrations and weak equivalences, have the left lifting property against fibrations, and dually, *trivial fibrations* have the right lifting property against cofibrations;

M5 the factorization systems factor maps into either a cofibration followed by a trivial fibration, or a trivial cofibration followed by a fibration.

We say an object $x \in \mathcal{C}$ is *cofibrant* (resp. *fibrant*) if the map from the initial object to x (resp. x to the terminal object) is a cofibration (resp. fibration).

Most of the well-behaved examples which come up in homotopy theory are build cellularly out of certain generating sets of morphisms.

Definition 1.2.3. Given a set of maps I, define a *relative I-cell complex* to be any transfinite composition of pushouts of elements of I; an object $x \in \mathcal{C}$ is an *I-cell complex* if the map $\varnothing \to x$ from the initial object is a relative I-cell complex. We denote the category of relative I-cell complexes by I-cell.

Definition 1.2.4. A model category $\mathcal{C}$ is *cofibrantly generated* if there exist sets of maps I and J such that a map f is a (trivial) fibration if and only if f has the right lifting property against all maps in I (resp. J).

Lemma 1.2.5. *If $\mathcal{C}$ is cofibrantly generated, then all (trivial) cofibrations are retracts of maps in I-*cell *(resp. J-*cell*).*

1.2.1 Building New Model Structures out of Old

It is often useful, possible, and necessary to place model structures on new categories $\mathcal{D}$ are related to another model category $\mathcal{C}$ by means of an adjunction $U : \mathcal{D} \leftrightarrow: \mathcal{C} : F$. We can in fact carry the model structure across after checking a significantly reduced number of conditions.

Theorem 1.2.6 ([Hir03, 11.3.2], Transfer Principle)**.** *Given a cofibrantly-generated model category $\mathcal{C}$ with generating (trivial) cofibrations I (resp. J), and an adjunction $U : \mathcal{D} \leftrightarrows \mathcal{C} : F$, let $FI = \{F(i) \mid i \in I\}$, and $FJ = \{F(j) \mid j \in J\}$. Then, if*

(1) the domains of FI and FJ are suitably "small", and

(2) U takes relative FJ-cell complexes to weak equivalences,

there $\mathcal{D}$ has a cofibrantly generated model structure with generating (trivial) cofibrations FI (resp. FJ), and weak equivalences those which get sent to one under U.

Remark 1.2.7. Some remarks:

(1) If the adjunction was *monadic* — that is, if $\mathcal{D} = \mathsf{Alg}_F(\mathcal{C})$ for some monad F — then the first condition can be replaced with just checking that $\mathcal{D}$ has coequalizers.

(2) This second condition above is satisfied if pushouts over maps in FJ are underlying trivial cofibrations.

In particular, this can be used to put model structures on all functor categories $\mathcal{V}^{\mathcal{D}}$ if $\mathcal{V}$ is cofibrantly generated:

Theorem 1.2.8 (e.g. [Hir03, 11.6.1])**.** *If $\mathcal{C}$ is a cofibrantly-generated model category, and $\mathcal{D}$ is any small category, then the diagram category $\mathcal{C}^{\mathcal{D}}$ has a cofibrantly-generated* projective *model structure, where weak equivalences and fibrations are detected levelwise.*

Given generating (trivial) cofibrations I (resp. J) of $\mathcal{C}$, the generating cofibrations in $\mathcal{C}^{\mathcal{D}}$ are

$$I^{\mathcal{D}} = \{\mathcal{D}(d, -) \cdot i \mid i \in I\}$$

and similarly for generating trivial cofibrations $J^{\mathcal{D}}$.

1.2.2 Semi-Model Categories

Sometimes, it will not be possible to obtain the full structure of a model category across a adjoint using the Transfer Principle; however, in more general scenarios, there is a weaker notion of a *semi*-model structure which will exist. Heuristically, in a semi-model structure:

(1) we can only factor maps into a trivial cofibration followed by a fibration if the domain is cofibrant, and

(2) only trivial cofibrations with cofibrant domains have the left lifting property against fibrations.

Specifically:

Definition 1.2.9. Suppose $\mathcal{C}$ is cofibrantly generated model category, $\mathcal{D}$ is complete and cocomplete, and we have an adjunction $U : \mathcal{D} \leftrightarrows \mathcal{C} : F$ such that U preserves small colimits.

We call $\mathcal{D}$ a *semi-model category* (or *J-semi-model category over $\mathcal{C}$*) if $\mathcal{D}$ has three classes of morphisms, again called *cofibrations, fibrations* and *weak equivalences*, such that:

(1) U preserves fibrations and trivial fibrations;

(2) $\mathcal{D}$ satisfies $M2$ (2-out-of-3) and $M3$ (retracts);

(3) cofibrations have the left lifting property against trivial fibrations, and trivial cofibrations with cofibrant domain have the let lifting property against fibrations;

(4) every map can be functorially factored into a cofibration followed by a trivial fibration, and every map with cofibrant domain can be functorially factored into a trivial cofibration followed by a fibration;

(5) the initial object in $\mathcal{D}$ is cofibrant; and

(6) fibrations and trivial fibrations are closed under pullback.

$\mathcal{D}$ is called *cofibrantly generated* if there exist sets of maps I and J in $\mathcal{D}$ such that a map is a (trivial) fibrations if and only if it has the right lifting property against I (resp. J), and the domains of I and J satisfy a "smallness" condition.

In particular, we note that semi-model structures have cofibrant replacements by Property (4).

Theorem 1.2.10 ([WY15, Theorem 2.2.2])**.** *Suppose $\mathcal{C}$ is a cofibrantly generated model category, with generating cofibrations I and trivial cofibrations J, and that we have a monadic adjunction $U : \mathsf{Alg}_F(\mathcal{C}) \leftrightarrow \mathcal{C} : F$ for some monad F. Further assume that, for any $F(I)$-cell* *complex $\mathcal{P}$, with cofibration $u : X \to Y$ and general map $h : X \to \mathcal{P}$ in $\mathcal{C}$, the pushout $\mathcal{P} \to \mathcal{P}[u]$ given by*

$$\begin{array}{ccc} F(X) & \xrightarrow{h} & \mathcal{P} \\ {\scriptstyle F(u)}\downarrow & & \downarrow \\ F(Y) & \longrightarrow & \mathcal{P}[u] \end{array}$$

is an underlying cofibration in $\mathcal{C}$, which is trivial whenever u is. Then $\mathsf{Alg}_F(\mathcal{C})$ has an induced cofibrantly-generated semi-model structure, with generating cofibrations $F(I)$ and trivial cofibrations $F(J)$, such that fgt *sends cofibrations with cofibrant domains to cofibrations.*

Here, we have restricted the condition from the Transfer Principle 1.2.6 to just considering the cases of pushouts of maps in FJ where the domain of the new map is cofibrant in $\mathcal{C}$ (specifically, in FI-cell). This weaker condition is matched by the weaker structure of a semi-model category.

See [Hov99, Spi01, Fre09, WY15] for more details.

Chapter 2

Operads and Trees

In this chapter, we will recall the definitions of both *operads* and *coloured operads*, the latter also known as *symmetric multicategories.* This second perspectives frames operads as a generalization of categories, where arrows are allowed to have multiple input sources; the original notion of operad will be a specific class of these operads.

2.1 Trees as Graphs

When working with operads, it will often be useful to visualize composition schema. It has been known since Boardman-Vogt [BV73] that, in this way, *trees* control the combinatorics of operads. We begin this chapter by introducing these objects geometrically.

Definition 2.1.1. A *graph* consists of a non-empty set E of *edges*, and a set V of tuples of edges, such that every edge belongs to at most two different vertices. Edges that belong to two distinct vertices are call *inner edges*, while others are *outer.* A graph is *connected* if for every pair of edges e and e', there exists a sequence of edges $\{e = e_0, e_1, \dots e_n = e'\}$ such that there exists vertices v_i with $v_i \supseteq \{e_{i-1}, e_i\}$.

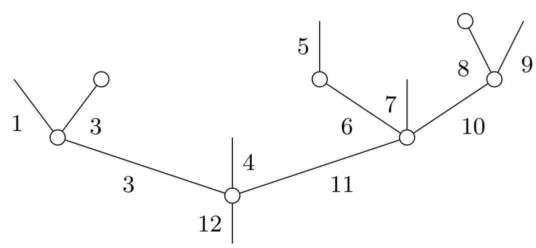
Definition 2.1.2. A *tree* is a finite connected graph with no loops and a chosen outer vertex call the *root.* The remaining outer edges are called *leaves*, the set of which is denoted $L(T)$.

We draw trees as graphs in the plane, directed downward, with the root at the bottom, and the leaves on top. Given a vertex v, denote by $in(v)$ the (possibly empty) set of input edges of v, and by t_v the output edge of v (so in the graph-theoretic notation, we have $v = in(v) \amalg \{t_v\}$). We denote the root edge and the root vertex by r.

Definition 2.1.3. A *leaf vertex* is a vertex with a whose input edges are all leaves. A *stump* is a vertex that only contains one edge; by definition of a tree, this means it only has an output edge, and no input edges.

Definition 2.1.4. The *degree* of a tree T, denoted $|T|$, is the number of vertices.

Example 2.1.5. The tree T below has four leaves (equivalently, $|L(T)| = 4$), two stumps, and six inner edges.



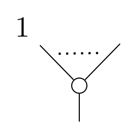
(2.1)

We highlight particular examples of trees.

Definition 2.1.6. The tree with a single edge and no vertex will be denoted η, and called a "stick". This is the only tree whose root is also a leaf.

Definition 2.1.7. The tree with a single vertex and n leaves will be denoted C_n and called the n-*corolla*.

η



C_n

Given two trees S and T and a leaf l of S, define the *grafting* of T on S along l, denoted $S \circ_l T$, is the graph with edges $E(T) \amalg E(S)/(r_T = l)$ and vertices $V(T) \amalg V(S)$.

This process can be iterated associatively. In particular, if we name the leaves of some n-corolla $L(C_n) = \{e_i\}$, and we have n different trees T_i, we denote the grafting of the T_i

along the i-th leaf of the n-corolla by $C_n(T_1, \ldots, T_n)$. Further, we note that the every tree can be written in this way:

Lemma 2.1.8. *Let T be a tree. If we name the non-root edges connected to the root edge $\{e_1, \ldots, e_n\}$, and denote by T_i the tree above (and including) the edge e_i, then $T = C_n(T_1, \ldots, T_n)$.* □

Planar Trees

Consider the tree from Example 2.1. We note that our depiction of it necessarily includes additional structure than just the edges and vertices: it includes a choice of *planarization*:

Definition 2.1.9. A *planarization* of a tree T is a choice of linear orderings of $in(v)$ for all vertices v. A *planar tree* is a tree T equipped with a planarization.

We note that a planarization of T actually encodes a total order on all of $E(T)$, not just each $in(v)$, by labeling up from the top-left:

Example 2.1.10. The labeling of the edges from Example 2.1 above denote their total ordering.

Further, the grafting of two (or more) planar trees $S \circ T$ is endowing with a unique induced planarization, such that the planar subtree $S \subseteq S \circ T$ is equal to the original planar tree S, and similarly for T.

2.2 Operads

Operads encode "generalized multiplications", by providing the data of "n-ary operations" for each n, as well as rules to compose these operations together.

Given a cocomplete closed symmetric monoidal category $\mathcal{V}$, we will define (single-coloured) $\mathcal{V}$-operads in three ways:

(1) with an explicit description;

(2) as monoids in a (asymmetric) monoidal category; and

(3) as algebras over a monad (Section 3.2).

Definition 2.2.1. Let Σ denote the *symmetric category*, the disjoint union of all the finite symmetric groups. Explicitly, objects are natural numbers $\{0, 1, 2, \ldots\}$, with

$$\Sigma(n, m) = \begin{cases} \Sigma_n & n = m \\ \varnothing & n \neq m \end{cases}$$

The category of *symmetric sequences* is the functor category $\mathcal{V}^{\Sigma^{op}} = \mathrm{Fun}(\Sigma^{op}, \mathcal{V})$.

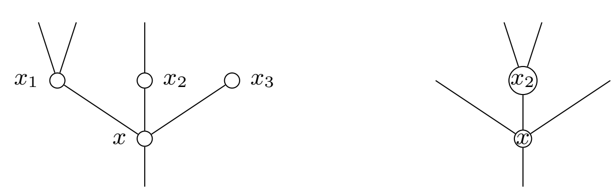
Remark 2.2.2. The "op" is not important now, as Σ is a groupoid. However, it will eventually makes constructions more consistent.

Definition 2.2.3 ([May72])**.** An *operad* in $\mathcal{V}$ is a collection $\mathcal{P} \in \mathcal{V}^{\Sigma^{op}}$ equipped with a unit map $\eta : I \to \mathcal{P}(1)$ and composition structure maps $\gamma : \mathcal{P}(n) \otimes \mathcal{P}(k_1) \otimes \ldots \otimes \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \ldots + k_n)$ which are unital , natural in n and $\{k_i\}$, and associative. A *map of operads* is a map of symmetric sequences which preserves the composition structure. We denote the category of $\mathcal{V}$-operads by $\mathcal{V}\mathsf{Op}_{\{*\}}$.

Remark 2.2.4. We can equivalently define an operad with a unit and "partial composition" structure maps $\gamma_i : \mathcal{P}(n) \otimes \mathcal{P}(k) \to \mathcal{P}(n + k - 1)$.

The results of these compositions can be represented by planar trees of height ≤ 2:

Example 2.2.5. The tree on the left below represents the image of $\gamma(x; x_1, x_2, x_3)$ with $x \in \mathcal{P}(3)$ and $x_1 \in \mathcal{P}(2)$, $x_2 \in \mathcal{P}(1)$, and $x_3 \in \mathcal{P}(0)$. Similarly, the tree on the right represents the image of $\gamma_2(x; x_2)$ with $x \in \mathcal{P}(3)$ and $x_2 \in \mathcal{P}(2)$::



(2.2)

Kelly [Kel05] and others (e.g. [Chi12]) state this in a more categorical way. The category of collections is equipped with two monoidal products, one of which is symmetric:

Definition 2.2.6. Let X and Y be symmetric sequences.

(1) The *tensor product* of the collections X and Y is the coend

$$X \otimes Y(-) := \int^{n,m\in\Sigma} X(n) \otimes Y(m) \otimes \mathsf{Iso}(n+m,-).$$

$\Sigma(0,-)$ is the unit of this operation, and it can be checked that this is symmetric and associative.

(2) The *composition product* of the collections X and Y is the coend

$$X \circ Y = \int^{n\in\Sigma} X(n) \otimes Y^{\otimes n}.$$

Here, $\Sigma(1,-)$ is the unit, but note that clearly this is *not* symmetric (though it remains associative).

Lemma 2.2.7 ([Kel05, Section 4])**.** *An operad is a monoid in* $(\mathcal{V}^{\Sigma^{op}}, \circ)$. □

Definition 2.2.8. An *algebra* over an operad $\mathcal{O}$ is some object $X \in \mathcal{V}$ with structure maps

$$\mathcal{O}(n) \otimes_{\Sigma_n} X^{\otimes n} \to X$$

which are associative and unital. Equivalently, X has a map $\mathcal{O} \circ X \to X$.

2.2.1 Examples

We provide some standard examples of operads:

Example 2.2.9. Given any $X \in \mathcal{V}$, the canonical operad is the *endomorphism operad* End_X for X, with $\mathsf{End}_X(n) = \mathcal{V}(X^{\otimes n}, X)$.

Example 2.2.10. The *commutative operad* $\mathsf{Comm}(n) = *$ for any n, where $*$ is the terminal object of $\mathcal{V}$. Dually, the *associative operad* $\mathsf{Assoc}(n) = \Sigma_n$ for any n. Algebras are commutative and associative monoids in $\mathcal{V}$, respectively.

Example 2.2.11. Specifically for $\mathcal{V} = \mathsf{Top}$, we say an operad $\mathcal{O}(n)$ is A_∞ if each space $\mathcal{O}(n)$ is contractible. Algebras over an A_∞-operad are "associative algebras up to coherent homotopy", and grouplike A_∞-algebras are equivalent to loop spaces $X = \Omega^1 Y$.

Example 2.2.12. Again for $\mathcal{V} = \mathsf{Top}$, we say an operad $\mathcal{O}$ is E_∞ if $\mathcal{O}(n) \simeq E\Sigma_n$; that is, if each space $\mathcal{O}(n)$ is Σ_n-free and contractible. Algebras are "commutative up to coherent homotopy". In [May72], May shows that grouplike E_∞-space are infinite loop spaces (and hence connective spectra); further in [EKMM97] that E_∞-spectra and commutative ring spectra (in any point-set model for spectra) are equivalent.

We should think of E_∞ operads as universal "homotopical deformations" of the commutative operads, in that E_∞-algebras X have maps from $X^n \to X$, which are not unique as with commutative monoids, but instead "unique up to all higher homotopies". This notion is made slightly more precise in Lemma 2.2.27 below.

Indeed, E_∞-algebras appear in many places in homotopy theory; see [May77] for more details.

Notation 2.2.13. As above, we refer to objects in TopOp as "topological operads". In a possible of notation, we refer to objects in sSetOp as simply "simplicial operads", and denote the category by sOp; we observe that this category is a full subcategory of the category $\mathsf{Op}^{\Delta^{op}}$ of "simplicial objects in operads".

2.2.2 Coloured Operads

Weiss07, BM07

In the above section, we restricted our attention to "single-coloured operads", as prepared by algebraic topologists. However, in many ways, allows for multiple colours is a more natural construction, as it demonstrates that operads are a generalization of the notion of a category; indeed, coloured operads are also called *symmetric multicategories*. See the introduction to [Wei07] for more on this perspective.

Heuristically, a *coloured operad* is a category where arrows are allowed to have multiple source objects. Explicitly, we start with a set $\mathfrak{C}$ of *colours* (or objects). Now, we define a *signature* to be a tuple $\xi = (c_1, \ldots, c_n; c)$ in $\mathfrak{C}^{\times n} \times \mathfrak{C}$ for some $n \in \mathbb{N}_0$.

Definition 2.2.14. The category of signatures, $\Sigma_{/\mathfrak{C}}$, is the Grothendieck construction on $\mathsf{Sig} : \Sigma \to \mathsf{Cat}$ sending n to $\mathfrak{C}^{\times n} \times \mathfrak{C}$. A $\mathfrak{C}$-*symmetric sequence* is a functor $\Sigma_{/\mathfrak{C}} \to \mathsf{Set}$.

Definition 2.2.15. A $\mathfrak{C}$-*coloured operad* is a $\mathfrak{C}$-symmetric sequence $\mathcal{O}$ with units $1_c \in \mathcal{O}(c; c)$ for each $c \in \mathfrak{C}$, and composition maps

$$\mathcal{O}(a_1, \ldots, a_n; a_0) \times \prod_i \mathcal{O}(b_i^1, \ldots, b_i^{k_i}; a_i) \to \mathcal{O}(b_1^1, \ldots, b_n^{k_n}; a_0)$$

which are associative, $\mathfrak{C}$-equivariant, and unital..

Definition 2.2.16. Algebras over a $\mathfrak{C}$-coloured operad are collections $X = \{X(c) \mid c \in \mathfrak{C}\}$ of objects, with associative and unital structure maps

$$\mathcal{O}(c_1, \ldots, c_n; c_0) \times X(c_1) \times \ldots \times X(c_n) \to X(c_0).$$

Example 2.2.17. Any symmetric monoidal category $\mathcal{C}$ induces a coloured operad, where the colours are the objects of $\mathcal{C}$, and operations in $\mathcal{O}(A_1, \ldots, A_n; A_0)$ are precisely maps $A_1 \otimes \ldots A_n \to A_0$ in $\mathcal{C}$.

Coloured operads can also encode relationships between objects:

Example 2.2.18. Given any $\mathfrak{C}$-coloured operad $\mathcal{O}$, there is a $\mathfrak{C} \amalg \mathfrak{C}$-coloured operad whose algebras are arrows $f : A \to B$ in which A and B are $\mathcal{O}$-algebras and f is a map of $\mathcal{O}$-algebras.

Similarly, there is a $\{m, l\}$-coloured operad Mod which encodes encodes a monoid $X(m)$ and a left $X(m)$-module $X(l)$. See [FMY09, 2.10,2.11] for more details.

Example 2.2.19. For a fixed set $\mathfrak{C}$, there is a coloured operad whose algebras are $\mathfrak{C}$-coloured operads; see [BM07] and [GV12] for details.

Remark 2.2.20. There is also a composition product description of coloured operads; see the Appendix of [BM07].

2.2.3 Model Structures on Operads

We would like to describe the homotopy theory of operads in a category $\mathcal{V}$, as it is often useful to have a model category to work with.

First, if $\mathcal{V} = \mathsf{Set}$, we have the following extension of the "folk" model structure of categories:

Theorem 2.2.21 ([Wei07, Theorem 1.6.2])**.** *There exists a model structure on set-operads where*

- *the weak equivalences are operadic equivalences (generalizing equivalences of categories);*
- *the cofibrations are maps $f : \mathcal{O} \to \mathcal{P}$ which are injective on colours;*
- *the fibrations are maps $f : \mathcal{O} \to \mathcal{P}$ such that $j^*f : j^*\mathcal{O} \to j{*}\mathcal{P}$ is a categorical fibration (where $j^* : \mathsf{Op} \to \mathsf{Cat}$ is the right adjoint to the inclusion of categories into operads).*

In the more general settings of $\mathcal{V}$-operads for general $\mathcal{V}$, there has been significant work in many directions to endow $\mathcal{V}\mathsf{Op}$ with the model structure. We give two key results; each source has additional references to similar works.

Definition 2.2.22. We say the category $\mathcal{V}\mathsf{Op}_{\mathfrak{C}}$ has the *projective* model structure if there exists a (necessarily unique) model structure where $f : \mathcal{O} \to \mathcal{P}$ is a weak equivalence or fibration if $\mathcal{O}(\xi) \to \mathcal{P}(\xi)$ is for all signatures ξ.

We have the following results:

Theorem 2.2.23 ([BM07, Theorem 2.1])**.** *If $\mathcal{V}$ is a cofibrantly generated monoidal model category with cofibrant unit and symmetric monoidal fibrant replacement functor, with a co-commutative co-associative co-algebra interval, then the category of $\mathfrak{C}$-coloured operads has the projective model structure.*

In another vein:

Theorem 2.2.24 ([WY15, Theorem 6.1.1])**.** *If $\mathcal{V}$ is a strongly cofibrantly generated monoidal model category such that for each $n \geq 1$ and each object $X \in \mathcal{V}^{\Sigma_n^{op}}$, the function $X \otimes_{\Sigma_n} (-)^{\square n} : \mathcal{V} \to \mathcal{V}$ preserves trivial cofibrations, then, in particular, the category of $\mathfrak{C}$-coloured operads has the projective model structure, where $\mathcal{O} \to \mathcal{P}$ is a weak equivalence or fibration whenever $\mathcal{O}(\xi) \to \mathcal{P}(\xi)$ is for all signatures ξ.*

These two results come from two different approaches on applying the Transfer Principle; the above uses Quillen Path Object Argument (e.g [BM03, 2.6]), while the latter approaches cellular extensions directly; see the discussion in Section 3.5. Other sources include, e.g. [Har09, Har10, PS15, Mur11].

We may also try put a model structure on all of $\mathcal{V}\mathsf{Op}$, for all $\mathfrak{C}$ simultaneously. We have the following:

Theorem 2.2.25 ([Cav14, Theorem 4.22])**.** *Let $\mathcal{V}$ be a cocomplete closed monoidal model category such that*

- *the unit is cofibrant,*
- *the model structure is right proper,*
- *there exists a set of generating $\mathcal{V}$-intervals,*
- *the class of weak equivalences is closed under transfinite composition; moreover, which is*
- *strongly cofibrantly generated*

- *contains a co-commutative co-associative interval object,*

then the category $\mathcal{V}\mathsf{Op}$ *of coloured* $\mathcal{V}$*-operads has a model structure with good properties, in particular extending the projective model structure above.*

See *loc. cite.* for the definitions used in the above.

Example 2.2.26. As particular cases of all of the above results, we have that the category of simplicial operads has the projective model structure.

Considering the model structure on operads in spaces, we have the following result recording a homotopical characterization of E_∞ operads:

Lemma 2.2.27 (Boardman-Vogt, Vogt, May, Berger-Moerdijk)**.** *If* $\mathcal{O} \to \mathsf{Comm}$ *is a cofibrant replacement of topological operads, then* $\mathcal{O}$ *is* E_∞*.*

2.3 The Category of Trees

Reference: [Wei07]

To fully exploit the combinatorial control of trees over operads, we will need a category of trees. As has been done, we will mention two equivalent approaches: one operadic, inspired by the geometry above, and one more algebraic.

These were both introduced in [MW07] and [Wei07].

2.3.1 Operadic Definition

This version highlights the fact that operads are generalizations of categories. In particular, we have a nerve-realization adjunction

$$\begin{tikzcd} \Delta \arrow[r, "\Delta^-"] \arrow[d, "Y"'] & \mathsf{Cat} \arrow[dl, shift left, "N"] \\ \mathsf{sSet} = \mathsf{Set}^{\Delta^{op}} \arrow[ur, shift left, "|-|"] & \end{tikzcd} \tag{2.3}$$

where Δ is the simplicial category of standard non-empty ordered finite sets and order-preserving set maps.

This captures the fact that strings of arrows controls all the combinatorics of compositions in categories. As compositions in coloured operads are controlled by trees, we expect this triangle to lift to a nerve-realization adjunctions into Op by extending the category Δ.

Definition 2.3.1. Given a tree T, let $\Omega(T)$ be the free coloured (symmetric) operad as follows: the set of colours is the set of edges, and morphisms are generated by $v \in \Omega(T)(e_1, \ldots, e_n; e)$, where $v \in V(T)$ is a vertex, e is the outgoing edge of v, and $in(v) = \{e_1, \ldots, e_n\}$ is the set of input edges.

Equivalently, given edges $a_1, \ldots, a_n, a$ of T, we have $\Omega(T)(a_1, \ldots, a_n; a) = \varnothing$ if there does not exist a subtree S of T which has leaves $\{a_i\}$ and root a; if such an S does exist, this equals Σ_n (as we have *symmetrized* the non-symmetric operad generated by T).

Definition 2.3.2. The *dendroidal category* Ω is defined to be the full subcategory of Op spanned by the $\Omega(T)$ for all trees T.

We observe that the image of the *linear* trees is precisely Δ, and we have a natural inclusion $l : \Delta \hookrightarrow \Omega$.

We will go over morphisms in more depth when discussing the broad poset formalism, but we give a preview of it here. All maps in Ω can be decomposed, uniquely up to isomorphism, into two types of maps: faces and degeneracies.

Definition 2.3.3. Given a tree T and a vertex $v = \{e, e'\} \in V(T)$, the operad $\Omega(T \setminus v)$ is missing the colour e' and the unary operation generated by v. We have a map $\sigma : T \to T \setminus v$ which on colors is the identity away from e and e' but sends both e and e' to the edge e, and on operations sends the unary operation $v \in \Omega(T)(e'; e)$ to $id_e \in \Omega(T \setminus v)(e; e)$ and is the identity on everything else.

Maps of this form are called *elementary degeneracy maps.*

Definition 2.3.4. Given a tree T and an inner edge e with input vertex v and output vertex u, the operad $\Omega(T/e) \in \Omega$ is missing the colour e and the generating operations $u \in \Omega(T)(in(u); e)$ and $w \in \Omega(T)(in(v); t_v)$, but instead has a new generating operation $s \in \Omega(T/e)(in(u) \amalg in(v) \setminus \{e\}\,; t_v)$. Further, we have a natural map $\varphi : T/e \to T$ which sends $s \to w \circ_e u$ and is the identity elsewhere.

Maps of this form are called *elementary inner face maps.*

Definition 2.3.5. Given a tree T and and an "outer cluster" C, there is an operad $\Omega(T\setminus C)$ which is missing the specified outer edges in C, and well as the operation generated by the vertex of C. Moreover, there is an obvious inclusion of operads $T \setminus C \to T$.

Maps of this form are called *elementary outer face maps.*

2.3.2 Broad Posets

To give ourselves more precision when discussing operations on trees, we recall the notion of a broad poset. Heuristically, broad posets are to posets as multicategories are to categories.

To begin, given a set E, let E^* denote the free abelian monoid generated by E. Explicitly, elements of E^* are unordered words, e.g. $\bar{a} = a_1a_2a_3a_4 = a_2a_4a_3a_1$ with each $a_i \in E$, and we write $a_i \in \bar{a}$ if a_i appears in the tuple.

Definition 2.3.6. A (commutative) *broad relation* is a set relation $\leq$ on (E^*, E). A (commutative) *broad poset* structure on E is a broad relation $\leq$ such that, for all $a, a_i, b \in E$ and $\bar{c}_i \in E^*$:

reflexivity $a \leq a$;

antisymmetry if $a \leq b$ and $b \leq a$ then $a = b$;

broad transitivity if $a_1a_2 \dots a_n \leq b$ and $\bar{c}_i \leq a_i$, then $\bar{c}_1 \dots \bar{c}_n \leq b$.

A map of broad posets is a set function which preserves broad relations, as expected.

As this notion will be used to model trees, we will refer to the elements of E as *edges.*

Definition 2.3.7. Given an edge $a \in E$, we denote the poset of *descendants* of a by $\widehat{a} := \{b \in E^* \mid b \lneq a\}$.

Definition 2.3.8. An edge $a \in E$ is called

(1) a *leaf* if $\widehat{a} = \varnothing$;

(2) a *stump* if $\widehat{a}$ is the monoid unit $\{\epsilon\}$;

(3) an *internal node* if $\widehat{a}$ is neither empty nor equal to $\{\epsilon\}$.

In either of the latter two cases, if $\widehat{a}$ has a maximum element, denote this element by $a^{\uparrow}$ and call it the *successor* of a; the elements of $a^{\uparrow}$ will be called the *children* of a.

Definition 2.3.9. We call a broad poset *simple* if, for any relation $a_1 \dots a_n \leq a$, $a_i = a_j$ implies $i = j$.

A broad poset structure on E also induces two preorders, one on E, and one on E^*:

Definition 2.3.10. Given $\bar{a}$ and $\bar{b} = b_1 \dots b_n$ in E^*, we write $\bar{a} \leq \bar{b}$ if we can write $\bar{a} = \bar{a}_1 \dots \bar{a}_n$ with each $\bar{a}_i \leq b_i$.

Definition 2.3.11. We say a is *dominanted* of b, written $a \leq_d b$, if there exists a broad relation $\bar{a} \leq b$ such that $a \in \bar{a}$. A $\leq_d$-maximum element, if it exists, is denoted r_E and called the *root* of E.

Definition 2.3.12. A *dendroidally ordered set* is a simple finite broad poset E which

(1) has a root, and

(2) each a is either a leaf, or it has a successor.

Definition 2.3.13. If E is a dendroidally ordered set, and $a \neq r_E$ is not the root, the *parent* of a, denoted $a_{\downarrow}$, is the unique edge such that $a \in (a_{\downarrow})^{\uparrow}$.

Definition 2.3.14. If E is dendroidally ordered, and $a \in E$ is any edge, the set of descendants has a minimum element, which we denote a^{λ}. Equivalently, a^{λ} is the set of $\leq_d$-minimal elements in the poset $\{b \in A \mid b \leq_d a\}$ of elements dominated by a.

Theorem 2.3.15 ([Wei07, Theorem 2.3.21])**.** *The category of dendroidally ordered sets with broad maps is equivalent to the category Ω.* $\square$

With this in mind, we will refer to either notion as simply *trees*, and will denote them by the letter T. To rectify notation and nomenclature between the two models:

(1) if v is a vertex, denote the output edge t_v; then the children of t_v are the input edges of v: $(t_v)^\uparrow = in(v)$.

(2) a vertex may be referred to by its name v, by the broad relation $t^\uparrow \leq t$ it generates, or its output edge t_v; if t is not a leaf, denote the vertex directly above t by v_t.

(3) if $a \in E(T)$ is not the root, the parent $a_\downarrow$ is equal to the outgoing edge of the vertex v_a.

(4) $\widehat{a}$ is the set of all nodes above v_a.

(5) a^λ is the set of "leaves above a"; that is, leaves of T whose leaf-root path contains a.

Definition 2.3.16. Let Ω_0 denote the core groupoid of Ω, the wide subcategory with only and all isomorphisms, and Υ the full subcategory of Ω_0 spanned by the corollas (equivalently, the broad posets with only one generating relation).

Υ is *almost* a full subcategory of Ω; however, it is missing the maps out of C_1 which factor through the degeneracy $C_1 \to \eta$.

Definition 2.3.17. We define the "valence" functor $val : \Omega \to \Upsilon$ by sending T to the broad poset $(L(T) \amalg \{r_T\}, \leq)$ with the single generating relation $L(T) \leq r_T$. Note that we have a canonical map $val(T) \to T$, which is an inclusion when T is not the stick η, and is a degeneracy for $T = \eta$.

Subtrees are the relevant subobjects in the category Ω.

Definition 2.3.18. A *subtree* S of T, written $S \subseteq T$, is a subset whose induced broad poset structure is again dendroidally ordered. A subtree $S \subseteq T$ is *maximal* if $|S| = |T| - 1$.

Definition 2.3.19. Given any edge $a \in E(T)$, we can form the descendancy subtree $T_{\leq_d a} \subseteq T$, the sub-broad poset of T spanned by those edges $b \in T$ with $b \leq_d a$. It is easy to check that this indeed forms a tree.

We can also define the "image" of a vertex, or more generally any relation, under a map of trees.

Definition 2.3.20. Given a map $f : S \to T$ of trees and a relation $\bar{e} \leq e$ in S, define the *subtree image* of the relation under f to be defined as follows. Define the edges to be the set

$$T_{f(\bar{e}\leq e)} = \left\{ t \in T \mid t \leq_d e \text{ and there exists } e' \in \bar{e} \text{ such that } t \geq_d e' \right\},$$

with relations all those $t^{\uparrow} \leq t$ with $\{t\} \amalg t^{\uparrow} \subseteq T_{f(\bar{e}\leq e)} \amalg \epsilon$, where ϵ is the monoid unit.

2.3.3 Morphisms

There are three generating classes of morphisms in Ω: isomorphisms, faces, and degeneracies. We begin with degeneracies.

Definition 2.3.21. An edge $e \in T$ is called a *only child* if $(e_{\downarrow})^{\uparrow} = \{e\}$; that is, the output vertex of e is unary. An *elementary degeneracy map* is a broad poset morphism of the form $\sigma_a : T \to T \setminus a$, where a is an only child, and $T \setminus a$ has the same generating relations as T, except replacing $a^{\uparrow} \leq a$ with $a^{\uparrow} \leq a_{\downarrow}$. If $a_1, \ldots, a_n$ are (simultaneous) only children of T, we denote by $\sigma_{a_1,\ldots,a_n}$ the composite map $T \to T \setminus \{a_1, \ldots, a_n\}$. A *degeneracy* is a composition of elementary degeneracies and isomorphisms.

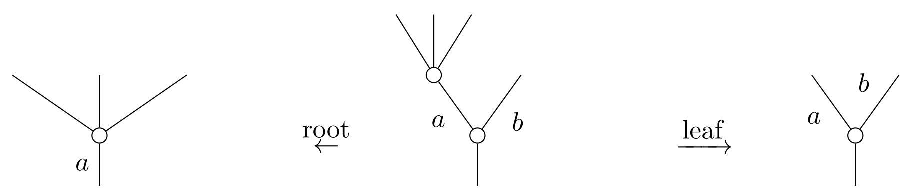
Degeneracies are surjective on edges, and moreover are the only non-injective maps. The injective (but not surjective) maps are called *faces*:

Definition 2.3.22. An *elementary face map* is an inclusion of the form $S \hookrightarrow T$, where S is a maximal subtree of T. Denote by $\Phi_1(T)$ the set of elementary face maps. A *face map* is a composite of elementary face maps and isomorphisms.

Face maps can be classified into two different types: inner and outer.

Definition 2.3.23. An *outer cluster* of a tree T is a collection of outer edges C such that $C = a^{\uparrow}$ for some edge a (in particular, C may be empty, as long as $a^{\uparrow}$ is the monoid unit). Denote by v_C An *elementary outer face map* is an inclusion of the form $\partial_C : T \setminus C \hookrightarrow T$, where C is an outer cluster and $T \setminus C$ no longer has the generating relation $a^{\uparrow} \leq a$. If $C_1, \ldots, C_n$ are (simultaneous) outer clusters, we denote by $\partial_{C_1,\ldots,C_n}$ the composite map $T \setminus C_1, \ldots, C_n \hookrightarrow T$. A *simple outer face map* is a composite of elementary outer faces; an *outer face map* is a composition of elementary outer face maps and isomorphisms.

Example 2.3.24. There are two types of outer clusters: a "leaf cluster" or a "root cluster". The following example gives a toy example with both types of clusters.



Example 2.3.25. The following are also outer face maps:

- the inclusion of any edge into a corolla;
- the inclusion of any descendancy subtree;
- the subtree image in T of any vertex $v \in V(S)$ over a map $f : S \to T$.

Lemma 2.3.26. *Given an injective map $f : S \to T$, the following are equivalent:*

(1) f is an outer face map;

(2) $S = T_{f(val(S))}$, where here $val(S)$ is the relation in S corresponding to the single generating relation in $val(S)$;

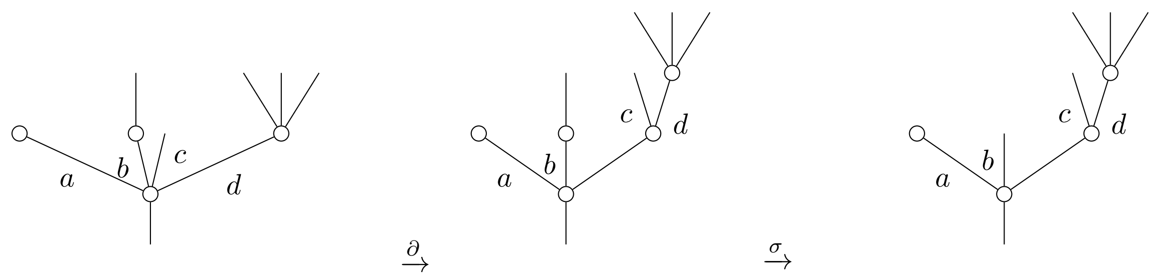
(3) if $\bar{a} \leq a$ is a relation in T with $\{a\} \amalg \bar{a} \in T_{f(val(S))}$, then $\bar{a} \leq a$ is a relation in S.

We now define the second type of face map.

Definition 2.3.27. An *elementary inner face map* is an inclusion of the form $\partial_e : T/e \hookrightarrow T$, where e is an inner edge of T, and T/e has the induced broad poset structure from T; that is, T/e has all generating broad relations except $e^{\uparrow} \leq e$. Composites of elementary inner face maps which remove the edges $e_1, \ldots, e_n$ will be denoted $T/\{e_1, \ldots, e_n\} \hookrightarrow T$.

A *simple inner face map* is a composite of elementary inner faces; an *inner face map* is a composite of elementary inner face maps and isomorphisms.

Example 2.3.28. The following is an inner face map followed by a degeneracy:



Remark 2.3.29. As notated above, if $T \in \Omega$ is not the stick η, then $val(T) \to T$ is equal to the inclusion of the smallest inner face of T. If $T = \eta$, then $val(\eta) = C_1$, and then the map $val(\eta) \to \eta$ is the degeneracy.

We note that simple outer faces and simple inner faces both form a poset:

Definition 2.3.30. The *outer face poset* (respectively, *inner face poset*), denoted $\mathsf{Out}(T)$ (respectively, $\mathsf{Inn}(T)$), has as objects all simple outer (inner) faces, thought of as subsets of $E(T)$, with the relation given by inclusion.

We note that $Q \leq R$ is a relation if and only if there is a simple outer (inner) face map $Q \to R$.

Interestingly, the outer face poset is *functorial* on T:

Lemma 2.3.31. *There is a functor* $\mathsf{Out} : \Omega \to \mathsf{Poset}$*, given by* $S \mapsto \mathsf{Out}(S)$*.*

Proof. Given a map of trees $f : S \to T$, and a subtree $S' \subseteq S$, define

$$\mathsf{Out}(f) = T_{f(val(S'))},$$

where by $val(S')$ we mean the relation in S given by the single relation in $val(S')$. □

A particular subposet of $\mathsf{Out}(T)$ will be of technical importance.

Definition 2.3.32. The *core poset* of a tree, denoted either $\mathsf{Out}_c(T)$ or $Sc(T)$, is the subposet of simple outer faces which are corollas or sticks.

This is equivalent to the poset of vertices and edges, with relations generated by $e \le v$ if e is connected to v.

Theorem 2.3.33 ([Wei12, Theorem 6.1])**.** *An elementary face map is either an elementary outer face or an elementary inner face map.* □

Theorem 2.3.34 ([MW07, Lemma 3.1])**.** *A map $\varphi : S \to T$ is a face map if and only if it is injective. Moreover, any face map $\varphi : S \to T$ has a decomposition, unique up to isomorphism, as $f : S \xrightarrow{\varphi_i} S' \xrightarrow{\varphi_o} T$ of an inner face φ_i followed by an outer face φ_o.* □

In general, isomorphisms can be tricky to state. However, we can use the following inductive description of a tree to produce an inductive description of the group of automorphisms of a tree.

Lemma 2.3.35. *For any $T \in \Omega$, let $e_1 \dots e_n \le r$ denote the root vertex of T, and let T_i be the broad poset $\{e \in T \mid e \le_d e_i\}$ (that is, the tree branch above e_i). Then T is obtained via grafting the T_i onto the root n-corolla, and we denote this decomposition by*

$$T = C_n \circ (T_1, \dots, T_n)$$

Lemma 2.3.36 ([BM03])**.** *Let $T \in \Omega$, and suppose T has a decomposition $T \simeq C_n \circ (T_1, \dots, T_n) = C_n \circ (T_1^1, \dots, T_{k_1}^1, T_1^2, \dots T_{k_r}^r)$ such that $T_j^i \simeq T_{j'}^{i'}$ iff $i = i'$. Then the group of* *automorphisms of T is isomorphic to*

$$\mathrm{Aut}(T) \cong \Sigma_{k_1} \wr \mathrm{Aut}(T_1^1) \times \ldots \times \Sigma_{k_r} \wr \mathrm{Aut}(T_r^1).$$

$\square$

Finally, we have a fundamental decomposition theorem for morphisms in Ω:

Theorem 2.3.37 ([Wei07, Theorem 2.2.6]). *Any map $f : S \to T$ has a decomposition, unique up to isomorphism, as $f : S \xrightarrow{\sigma} S' \xrightarrow{\varphi} T$ of a degeneracy σ followed by a face map φ.*

We specify another type of map, which will become useful later.

Definition 2.3.38. A map $\varphi : S \to T$ is *tall* if $\varphi(r_S) = r_T$ and $\varphi(L(S)) = L(T)$. We denote by Ω_t the wide subcategory of trees and tall maps.

Lemma 2.3.39. *A map $\varphi : S \to T$ is tall if and only if $\varphi(r_s) = r_T$ and φ restricts to a bijection $\varphi : L(S) \to L(T)$.*

Proof. It suffices to prove the "only if" statement, as the "if" is clear. Suppose l_1 and l_2 in $L(S)$ have the same image in $L(T)$. Then the relation $r_S^\lambda \leq r_S$ maps to $\varphi(r_S^\lambda) \leq \varphi(r_S) = r_T$ with $\varphi(r_S^\lambda) \subseteq L(T)$. However, since T is a simple broad poset, we conclude that all entries of $\varphi(r_S^\lambda)$ must be distinct, and hence $l_1 = l_2$. $\square$

These morphisms will be put to heavy use in Chapter 3.

2.3.4 Planar Trees

It will be technically convenient to have a description of Ω which includes planarizations of trees. Geometrically, this is equivalent to specifying a planarization of the tree T before generating $\Omega(T)$, and remembering this choice. In terms of broad posets, this can be stated as follows:

Definition 2.3.40 (Pereira). A *planarization* of a tree $T \in \Omega$ is an extension of the partial order $(E(T), \leq_d)$ to a total order $\leq_p$ such that, for all edges $a, b, c \in E(T)$, if $a \leq_p b$ while

$a \not\leq_d b$, then $b \geq_d c$ implies $a \leq_p c$ (note the appearance of a discrepency between the directions of these relations).

Proposition 2.3.41. *The above notion corresponds to the geometric notion from Section 2.1. Explicitly, there is a bijection of sets*

$$\{planarizations \leq_p \ of\ T\} \longrightarrow \coprod_{v \in V(T)} \mathsf{Iso}([n], in(v))$$

betweens planarizations of T and choices of total orderings of the input edges for each vertex of T. □

Definition 2.3.42. Let $\mathbb{T}$ denote a choice of category of planar trees and non-planar morphisms, such that there is exactly one representative of each planarization — that is, the only planar isomorphisms are the identity. Let $\mathbb{T}_0$ (respectively, $\mathbb{T}_t$) denote the wide subcategories of isomorphisms (respectively, isomorphisms and planar tall maps).

Remark 2.3.43. Our category $\mathbb{T}_0$ is equivalent to the category denoted $\mathbb{T}$ from [BM03].

Lemma 2.3.44. *Let S and T be planar trees, and C a planar corolla.*

(1) If there exists a planar tall map $f : S \to T$, then f is the unique *planar tall map from $S \to T$.*

(2) $C = val(T)$ if and only if there exists a planar tall map $C \to T$.

□

We let Σ denote the *symmetric category* $\Sigma = \coprod_n \Sigma_n$, the disconnected category with objects $\mathbb{N}$ and morphisms $\Sigma(n, n) = \Sigma_n$.

Lemma 2.3.45. *The full subcategory of $\mathbb{T}$ spanned by the (planar) corollas is equal to Σ.* □

We note that the categories Ω and $\mathbb{T}$ (respectively Ω_0 and $\mathbb{T}_0$) are equivalent. However, the latter categories provide slightly more structure, which allows us to make canonical categorical constructions.

Definition 2.3.46. Define the functor $C_{(-)} : \Sigma \to \mathbb{T}$ by sending n to the (planar) n-corolla.

Definition 2.3.47. Given $T \in \mathbb{T}$, we denote also by val the planarized valence functor $val : \mathbb{T}_t \to \Sigma$ by $T \mapsto val(T)$, where $val(T)$ is the (now) totally ordered set $(L(T) \amalg \{r_T\}, \leq_p)$. This provides a natural group homomorphism $\mathrm{Aut}(T) \to \Sigma_{|L(T)|}$ for all trees T.

Locally, given a vertex $v = (e^{\uparrow} \leq e) \in V(T)$, define T_v to be either (1) the totally ordered set $(e^{\uparrow} \amalg \{e\}, \leq_p)$, or (2) the totally ordered set $(e^{\uparrow}, \leq_p)$. Note that any automorphism φ on T induces canonical elements in $\Sigma_{|in(v)|} = \mathrm{Aut}(T_v) = \mathrm{Aut}(T_{\varphi(v)})$ for each $v \in V(T)$.

Remark 2.3.48. The two observations, that we have *canonical* homomorphisms $\mathrm{Aut}(T) \to \Sigma_{|L(T)|}$ and $\mathrm{Aut}(T) \to \prod_{v \in V(T)} \Sigma_{|in(v)|}$, underly much of the benefit of having planar structures around.

Abusing notation further, we will also denote by val the restriction of the above functor to $\mathbb{T}_0$.

Remark 2.3.49. We observe that $val : \mathbb{T}_0 \to \Sigma$ is not a Grothendieck construction, as for most trees T with n-leaves, the homomorphism $\mathrm{Aut}(T) \to \Sigma_n$ is not surjective: e.g. $T = C_2 \circ (C_3, C_4)$ has automorphism group $\Sigma_3 \times \Sigma_4$.

We could force this map to be one, as done in [BM03], by additionally equipping our planar trees with an independent ordering of the leaves. However, this additional structure would not be compatible with some of the later constructions, and moreover, we can prove most of the desired properties that being a Grothendieck fibration could give us by direct methods.

2.4 Dendroidal Sets

Reference: [Wei07, CM11, CM13a, CM13b, MW09]

Dendroidal sets, introduced by Moerdijk-Weiss [MW07, MW09], generalize simplicial sets, and provide a combinatorial analysis of operads parallel to the combinatorial analysis of categories provides by sSet. As a large part of this thesis will be generalizing this framework, in this Section we will highlight the major definitions and results.

Recall from Section 2.3 that Ω denotes the category of dendroidally ordered broad posets, or equivalently the full subcategory of coloured operads spanned by the free operads $\Omega(T)$ ranging of all trees T.

Definition 2.4.1. The category dSet of *dendroidal sets* is the category of presheaves $\mathsf{Set}^{\Omega^{op}}$. Given a tree T, let $\Omega[T]$ denote the representable presheaf $\mathsf{dSet}(-, \Omega(T))$.

Given any dendroidal set $X \in \mathsf{dSet}$, elements of $X(T)$ are called *dendrices* (or a *dendrix*), and are uniquely determined by a characterizing map $\Omega[T] \to X$ of dendroidal sets. We say a dendrix $x \in X(T)$ is *degenerate* if the characterizing map factors through some $\Omega[\sigma] : \Omega[T] \to \Omega[S]$ with σ a degeneracy in Ω; otherwise, we say x is *non-degenerate*.

We list some of the main properties:

Proposition 2.4.2. *The following hold:*

(1) $\mathsf{dSet} \downarrow \Omega[\eta]$ *is canonically equivalent to* Δ*, and we have an adjunction*

$$i_! : \mathsf{sSet} \rightleftarrows \mathsf{dSet} : i^*$$

(2) The functor $i_!$ *is fully-faithful (with* sSet *the subcategory of linear trees), and only linear trees can map to linear trees*

(3) We have a nerve-realization adjunction, generalizing Diagram (2.3)

$$\begin{tikzcd} \Omega \arrow[r, "\Omega(-)"] \arrow[d, "\Omega[-]"'] & \mathsf{Op} \arrow[dl, shift left, "N_d"] \\ \mathsf{dSet} \arrow[ur, shift left, "\tau_d"] & \end{tikzcd} \tag{2.4}$$

where $N\mathcal{O}$ is called the dendroidal nerve of $\mathcal{O}$, and $\tau(X)$ is the operad generated by X.

We observe that any map $T \to T'$ of trees induces a map $\Omega[T] \to \Omega[T']$ between the representable presheaves. Moreover, if there original map were a monomorphism, the induced map would be as well. Now, given a face map $\varphi : S \to T$, let $\partial_\varphi \Omega[T]$ be the image in $\Omega[T]$ of the induced map $\Omega[\varphi] : \Omega[S] \to \Omega[T]$; if φ is an elementary inner face map $T/e \to T$, we denote it by ∂_e. We define the *boundary* of $\Omega[T]$ inclusion, denoted

$$\partial\Omega[T] \hookrightarrow \Omega[T],$$

to be the colimit over the inner face poset of these $\partial_\varphi \Omega[T]$.

If e is an inner edge of T, we define the colimit of all faces *except* ∂_e to be the *inner horn* associated to e, denoted

$$\Lambda^e[T] \hookrightarrow \Omega[T].$$

Further, we can a map *inner anodyne* if it is in the saturation of the set of inner horn inclusions under retracts of transfinite compositions of pushouts.

This precisely generalizes the notions of boundaries, horns, and anodyne maps of simplicial sets; in particular, if $T = [n]$ were linear, then $\partial\Omega[n] = \partial\Delta[n]$ and $\Lambda^i[T] = \Lambda^i[n]$.

Definition 2.4.3. We say a dendroidal set $X \in \mathsf{dSet}$ is a (dendroidal) *inner Kan complex*, or ∞-*operad*, if for all T and any inner edge e of T, X has the right lifting property against the inner horn inclusion $\Lambda^e[T] \hookrightarrow \Omega[T]$. Equivalently, the map

$$X(T) = \mathsf{dSet}(\Omega[T], X) \to \mathsf{dSet}(\Lambda^e[T], X)$$

is surjective. It is additionally called *strict* if this map is a bijection, or equivalently the lifts are *unique*.

We denote by Kan and SKan the full subcategories spanned by (strict) inner Kan complexes.

More generally, we call a map an *inner fibration* if it has the right lifting property against all inner horn inclusions.

It is immediate that the nerve $N\mathcal{O}$ of an operad is a strict inner Kan complex, agreeing with the naming convention of ∞-operads as "weak operads". More, the converse of this statement is true:

Proposition 2.4.4 ([MW09, Proposition 5.3, Theorem 6.1]). *A presheaf $X \in \mathsf{dSet}$ is a strict inner Kan complex if and only if $X \simeq N\mathcal{O}$ for some operad $\mathcal{O}$.*

This is shown by constructing a "homotopy operad" $Ho(X)$ for any inner Kan complex X, and the following results:

Proposition 2.4.5 ([MW09, Proposition 6.6, 6.10]). *For any inner Kan complex X,*

(1) there is a natural map $X \to N(Ho(X))$, which is an isomorphism if X is strict; and

(2) there is a natural map $\tau(X) \to Ho(X)$, which is an isomorphism of operads.

$\square$

We briefly discuss the construction of the homotopy operad $Ho(X)$; the full details can be found in [MW09]. First, we define the set of colours of $Ho(X)$ to be the set $X(\eta)$. Now, given $f, g \in X(C_n)$, we say that f is *homotopic* to g, written $f \simeq g$, if there is some $\gamma \in X(C_n \circ_e C_1)$ such that

- the root cluster outer face $\partial_r \gamma$ is degenerate;
- the leaf cluster outer face $\partial_l \gamma = f$; and
- the inner face $\partial_e \gamma = g$;

where e is any edge of C_n. It can be shown that this is independent of the choice of edge e, and forms an equivalence relation on $X(C_n)$.

Now, given $f \in X(C_n)$ and $g \in X(C_m)$, and a leaf $e \in C_n$, we say h *observes* the composition $f \circ_e g$, written $h \simeq f_e g$, if there exists a dendrix $\chi \in X(C_n \circ_e C_m)$ with

- the root cluster outer face $\partial_r \gamma = g$;

- the leaf cluster outer face $\partial_l \gamma = f$; and
- the inner face $\partial_e \gamma = h$.

It can be shown that this is transitive under the homotopy operation, in that if h and h' both observe the same composition, they are homotopic, and that the compositions of homotopic dendrices are homotopic.

Definition 2.4.6. Given colors $a_1, \ldots, a_n, a_0 \in X(\eta)$, define $X(a_1, \ldots, a_n; a_0) \subseteq X(C_n)$ to be those dendrices x such that $(e_i)^* x = a_i$, where $e_i : \eta \to C_n$ is the inclusion of the edge e_i (and e_0 is the root).

Define $Ho(X)(a_1, \ldots, a_n; a_0) := X(a_1, \ldots, a_n; a_0)/\sim$ to be the set of equivalences classes under the homotopy relation. Then, by the above discussion, the composition $[f] \circ_e [g] = [h]$ with $h \simeq f \circ_e g$ determines a well-defined operadic structure on $Ho(X)$.

We give another model for the image of the nerve functor.

Definition 2.4.7. The *Segal core* of a tree, denoted $Sc[T]$, is the union $Sc[T] = \bigcup_{Sc(T)} \Omega[C_{T_v}]$ over the images of the inclusion of each vertex corolla (where we recall $Sc(T) := \mathsf{Out}_c(T)$). Equivalently, $Sc[T] = \Lambda^I[T]$ is the smallest inner face of T.

Lemma 2.4.8 ([CM13a, Corollary 2.6])**.** *$X \in \mathsf{dSet}$ is isomorphic to some $N\mathcal{P}$ if and only if X is uniquely determined by its evaluation on corollas:*

$$X(T) = \mathrm{Hom}(\Omega[T], X) \simeq \mathrm{Hom}(Sc[T], X) \simeq \prod_{v \in V(T)} X(T_v)/\sim .$$

2.4.1 Model Structure and the Homotopy Coherent Nerve

The series of papers [CM11, CM13a, CM13b] showed that the homotopy theory of dendroidal sets was equivalent to that of simplicial operads, again paralleling the Quillen equivalence between quasi-categories and simplicial categories. Moreover, in these works, they generalize *all* the models for $(\infty, 1)$-categories (e.g. [Ber17] into the dendroidal-operadic setting.

In particular, they build a model structure on dSet such that ∞-operads are the fibrant objects. We briefly describe this structure.

Definition 2.4.9. A monomorphism $f : X \to Y$ in dSet is call *normal* if for all $T \in \Omega$ and non-degenerate $y \in Y(T) \setminus X(T)$, the stabilizer $\mathrm{Stab}_{\mathrm{Aut}(T)}(y)$ is trivial.

Theorem 2.4.10 ([CM11, Theorem 2.4]). *The category* dSet *can be endowed with a left proper cofibrantly generated model structure such that*

(1) cofibrations are the normal monomorphisms;

(2) anodyne extensions are trivial cofibrations;

(3) ∞*-operads are the fibrant objects;*

(4) fibrations $f : X \to Y$ *between* ∞*-operads are inner fibrations such that the map on categories induced by* f *is a categorical fibration;*

(5) the weak equivalences are the smallest class containing the inner anodyne extensions and the trivial fibrations which is closed under 2-out-of-3.

Theorem 2.4.11 ([CM11, Corollary 6.17]). *The adjoint* $\tau : \mathsf{dSet} \leftrightarrows \mathsf{Op} : N$ *is a Quillen pair.*

Now, restricting to simplicial operads, we Boardman-Vogt W-construction extends to a nerve-realization adjunction

$$\begin{tikzcd} \Omega \arrow[r, "W(-)"] \arrow[d] & \mathsf{sOp} \arrow[dl, shift left, "hc_N"] \\ \mathsf{dSet} \arrow[ur, shift left, "W_!"] & \end{tikzcd}$$

such that $W_! hc_N$ precisely recovers the W-construction.

The series of papers [CM11, CM13a, CM13b] mentioned above culminate in proving that this adjunction is a Quillen equivalence. They do not show this directly; instead, as mentioned above, they pass through passes through many intermediate stages: dendroidal Segal spaces, Segal operads, pre-operads, etc. While this thesis will note elaborate further on this aspect of the dendroidal sets story, the framework developed here provides a baseline to begin the equivariant analysis of the above work of Cisinski-Moerdijk in the future, by specifically providing good candidates for the various intermediate stages, as well as suggesting a categorical definition of W via Kan extensions.

We note that independent analysis of equivariant dendroidal sets and equivariant Segal spaces by Bergner-Gutierrez and Bergner-Chadwick is in progress.

Chapter 3

Categorical Constructions on Single-Coloured Operads

In this chapter, we will rebuild many categorical constructions on operads, including the coproduct of free operads, the coproduct with a free operad, and cellular extensions. Our discussion is organized such that we can easily construct a filtration on (in particular) cellular extensions, simply by creating a filtration on some indexing category (of "structured" trees). Much of this is inspired and built off of the work of Berger-Moerdijk in [BM03], as well as the organization schema of [Per16].

Much of the discussion below almost certainly generalizes in its entirety to coloured trees and multicoloured operads. However, as the notation is significantly more cumbersome in that case (cf. [WY15]), and since we did not find that it provided any additional clarity, we will restrict ourselves to the analysis of operads with a single colour.

To that end, we fix a cocomplete closed symmetric monoidal category, and we will write "(symmetric) sequence" to mean an object in $\mathcal{V}^{\Sigma^{op}}$, and "operad" to mean an object in $\mathcal{V}\mathsf{Op}^G_{\{*\}}$.

3.1 Nerve Evaluation Maps

We begin by building some of the categorical framework which provides us with the tools to study operads. First, we define an functor which synthesizes the dendroidal nerve and the underlying symmetric sequence.

Notation 3.1.1. We let F denote the category of *totally ordered* finite sets, and F_0 the subgroupoid of finite sets and isomorphisms (i.e. Δ and Σ are categorical skeleta of F and F_0).

Definition 3.1.2. Given any category $\mathcal{C}$, consider the functor $\mathsf{F}^{op} \to \mathsf{Cat}$ which sends a finite set A to the functor category $\mathcal{C}^A$, and denote by $\mathsf{F}\wr\mathcal{C} \to \mathsf{F}$ the associated Grothendieck fibration (see 1.1.9). Explicitly, objects of $\mathsf{F}\wr\mathcal{C}$ are pairs $(A, D : A \to \mathrm{Ob}(\mathcal{C}))$, and morphisms $(A, D) \to (A', D')$ are pairs of maps (f, φ), where $f : A \to A'$ is a map of sets, and $\varphi : D \to f^*D'$ is an arrow in $\mathcal{C}^{A'}$.

This construction also works for any subcategory of F, namely F_0.

Unpacking the definitions, the following is immediate.

Definition 3.1.3. Given a symmetric monoidal category $\mathcal{V}$, there is a natural map $\otimes : (\mathsf{F}_0 \wr \mathcal{V}^{op})^{op} \to \mathcal{V}$ which sends a tuple of elements in $\mathcal{V}$ to their (unordered) symmetric monoidal product.

Remark 3.1.4. If $\mathcal{V}$ were in fact *Cartesian* monoidal, then this functor would naturally extend to all of $(\mathsf{F} \wr \mathcal{V}^{op})^{op}$.

This operation is "associative" in the following sense:

Definition 3.1.5. Let $\delta^1 = \mathrm{coll} : \mathsf{F}_0{}^{\wr 2}\mathcal{V} \to \mathsf{F}_0 \wr \mathcal{V}$ be the functor which sends takes an object $(A, a \mapsto (B_a, (a, b) \mapsto x_{a,b}))$ to $(\amalg_A B_a, (a, b) \mapsto x_{a,b})$.

Lemma 3.1.6. *There is a natural "re-collating" or "re-partitioning" isomorphism $\Phi_\otimes$ as below:*

$$\begin{tikzcd}
\mathsf{F}_0{}^{\wr 2}\mathcal{V} \arrow[r, "\mathsf{F}_0\wr\otimes"] \arrow[d, "\mathrm{coll}"'] & \mathsf{F}_0 \wr \mathcal{V} \arrow[d, "\otimes"] \\
\mathsf{F}_0 \wr \mathcal{V} \arrow[r, "\otimes"] & \mathcal{V}
\end{tikzcd}$$

$\square$

Additionally, we have $\sigma^0, \sigma^1 : \mathsf{F}_0 \wr \mathcal{V} \to \mathsf{F}_0^{\wr 2}\mathcal{V}$ sending $(A, (v_a))$ to $(*, (A, (v_a)))$ and $(A, (*, v_a))$ respectively (in fact, with these maps, the iterates $\mathsf{F}_0^{\wr n}\mathcal{V}$ form a sort of weak *simplicial* object in categories, with the weakness encoded in these re-collating isomorphisms).

Furthermore, every sequence $Y \in \mathcal{V}^{\Sigma}$ induces a natural functor $Y^{\otimes} : (\mathsf{F}_0 \wr \Sigma)^{op} \to \mathcal{V}$ given by the composition

$$Y^{\otimes} : (\mathsf{F}_0 \wr \Sigma)^{op} \xrightarrow{\mathsf{F}_0 \wr Y} (\mathsf{F}_0 \wr \mathcal{V}^{op})^{op} \xrightarrow{\bigotimes} \mathcal{V}.$$

Explicitly, this is given by $Y^{\otimes}(A, D) = \bigotimes_{d \in D} Y(A(d))$.

Planar trees provide salient examples of elements of functors to $\mathsf{F} \wr \Sigma$:

Definition 3.1.7. Given $T \in \mathbb{T}$, recall that $V(T)$ denotes the set of vertices of T, while for each $v \in V(T)$, $T_v = T_{e^{\uparrow} \leq e}$ is the corolla $e^{\uparrow} \amalg e$ surrounding v; abusing notation, we will also let T_v just denote the set $e^{\uparrow}$, ordered by the planar structure.

Define the *vertex* functor $\mathbb{V} : \mathbb{T}_0 \to \mathsf{F}_0 \wr \Sigma$ by $T \mapsto (V(T), val)$ with $val : V(T) \to \Sigma$ denoting the map $v \mapsto T_v$.

Definition 3.1.8. Given a sequence Y, the *nerve evaluation* functor, denoted by $N_Y : \mathbb{T}_0^{op} \to \mathcal{V}$, is defined as the composite

$$\mathbb{T}_0^{op} \xrightarrow{\mathbb{V}} (\mathsf{F} \wr \Sigma)^{op} \xrightarrow{\mathsf{F}_0 \wr Y} (\mathsf{F}_0 \wr \mathcal{V}^{op})^{op} \xrightarrow{\otimes} \mathcal{V}.$$

N_Y can also be defined inductively (c.f. the appendix of [BM03]) by letting $N_Y(\eta) = I$, and for $T = t_n(T_1, \ldots, T_n)$, define

$$N_Y(T) = Y(n) \otimes N_Y(T_1) \otimes \ldots \otimes N_Y(T_n).$$

On morphisms, N_Y is induced by the Σ-action on Y and the symmetry isomorphisms in $\mathcal{V}$.

Remark 3.1.9. If $\mathcal{V} = \mathsf{sSet}$, and $Y \in \mathsf{sOp}$ is a simplicial operad, then is it easy to check that N_Y is isomorphic to the image of Y under the composite

$$\mathsf{sOp} \hookrightarrow \mathsf{Op}^{\Delta^{op}} \xrightarrow{(N)^{\Delta^{op}}} \mathsf{s}(\mathsf{dSet}) \simeq \mathsf{d}(\mathsf{sSet}) \xrightarrow{\text{fgt}} \mathsf{sSet}^{\mathbb{T}_0^{op}}.$$

If Y is actually an *operad*, then N extends to $\mathbb{T}_t$, the category of planar trees and planar tall maps, acting as the composition structure map on inner faces, and the inclusion of the unit on degeneracies. More, we have the following:

Lemma 3.1.10. *For any operad $\mathcal{P}$, $\mathcal{P}$ is isomorphic to the left Kan extension*

$$\begin{tikzcd} \mathbb{T}_t^{op} \arrow[r, "N_{\mathcal{P}}"] \arrow[d, "val"'] & \mathcal{V} \\ \Sigma^{op} \arrow[ur, "\mathrm{Lan}_{val} N"'] & \end{tikzcd}$$

Proof. Let $\mathcal{Q} = \mathrm{Lan}_{val} N$. Then $\mathcal{Q}(n) \cong \mathrm{colim}_{\mathbb{T}_t^{op} \downarrow n} N_{\mathcal{P}}(T)$. Then again composition is well-defined by grafting of trees, with unitality enforced by degeneracies. But we also have that $\Sigma^{op} \downarrow n$ is final in $\mathbb{T}_t^{op} \downarrow n$, so in fact

$$\mathcal{Q}(n) \cong \operatorname*{colim}_{\mathbb{T}_t^{op} \downarrow n} N_{\mathcal{P}}(C_n) \cong \operatorname*{colim}_{\sigma \in \Sigma_n} \mathcal{P}(n) \times \{\sigma\} \cong \mathcal{P}(n).$$

□

3.1.1 Pullbacks, Assembly, and Planar Tall Maps

Our general methodology for presenting the constructions in this chapter will to be to assemble categories of "structured trees", and exploit this structure to encode the information we aim to investigate.

We begin with the following definition:

Definition 3.1.11. Let $\mathbb{T}_1$ be the following pullback category:

$$\begin{tikzcd} \mathbb{T}_1 \arrow[r, "\mathbb{V}"] \arrow[d, "d_1"'] & \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[d, "\mathsf{F}_0 \wr val"] \\ \mathbb{T}_0 \arrow[r, "\mathbb{V}"] & \mathsf{F}_0 \wr \Sigma \end{tikzcd}$$

Explicitly, objects are trees along with "assembly data". Explicitly, each vertex T_v is equipped with a tree S^v such that $val(S^v) = T_v$.

By Lemma 2.3.44, this is the same data as a collection of planar tall maps $T_v \to S^v$, one for each vertex $v \in V(T)$.

Equivalently, we have a tree T equipped with a functor $S^{(-)} : \mathsf{Out}_c(T) \to \mathbb{T}$ and a natural transformation $U : id \Rightarrow S^{(-)}$ consisting of planar tall maps.

We call this "assembly data", because there is a second natural map $\mathbb{T}_1 \to \mathbb{T}_0$, which assembles these trees S^v together to form a new tree:

Proposition 3.1.12 (Pereira)**.** *Given $(T, (S^v)) \in \mathbb{T}_1$, the colimit* $\mathrm{colim}_{V(T)} S^{(-)}$ *in the category $\mathbb{T}$ exists, and is denoted $T \wedge (S^v)$.*

Explicitly, $T \wedge (S^v)$ is the broad poset with underlying set the colimit of the underlying sets, and broad relations generated by the S^v.

Heuristically, $T \wedge (S^v)$ "inserts" the tree S^v at the node T_v of T, where the planar tall map identifies the outer edges of S^v with the edges of T_v. We highlight that if T_v is unary and S^v is a stick, this *deletes* the node v.

Proof. Since vertices are determined by their output edge, we can write assembly data as a tuple $(T, (S^t))$ where now t ranges over all non-leaf edges of T.

A priori, we only know that the colimit is a "pre-broad poset": it is a set with a broad relation $\leq$ which is reflexive and transitive. We need to first check that $\leq$ is antisymmetric, and then that it is dendroidally ordered.

We begin by characterize the underlying set $\amalg_{V(T)} V(S^v)$ of $T \wedge (S^v)$. We note that in this set, every inner edge t of T identifies the root of S^t with a particular leaf of $S^{t_\downarrow}$, where we recall that $t_\downarrow$ is the unique edge such that $t \in (t_\downarrow)^\uparrow$. Thus, two distinct edges of T can only be identified if there exists some $S^{t'}$ which has a leaf that is also a root; that is, if there exists an $S^{t'}$ which is a stick. However, this implies, in particular that $v_{t'}$ is unary, and hence we can consider the degeneracy $T \to T \setminus \{v_{t'}\}$. The assembly data $(T, (S^t))$ forgets to an assembly data $(T \setminus \{v\}_{t'}, (S^t))$, and moreover these have the same colimit. Thus, without loss of generality, we may assume that no S^t are sticks, and hence, for an edge t in T, the complete set of relations in the underlying set of $T \wedge (S^t)$ is given by

$$[t] = \begin{cases} [l_{t_\downarrow}] = [r_t] & t \text{ an inner edge of } T \\ [l_{t_\downarrow}] & t \text{ a leaf of } T \\ [r] & t \text{ the root of } T \end{cases}$$

where l is a leaf of $S^{t_\downarrow}$, r is the root of S^t, and $[e]$ is the equivalence class of the edge e in the set of edges in $T \wedge (S^t)$.

Hence, we can write each edge of $T \wedge (S^t)$ uniquely as (e, t), with $e \in S^t$ not a root unless $v_t \in V(T)$ is the root vertex. Further, there is at most one generating relation $(e, t)^\uparrow \leq (e, t)$, where the tuple of edges $(e, t)^\uparrow$ is defined as

$$(e,t)^\uparrow = \begin{cases} (e^\uparrow, t) & e \text{ is not a leaf in } S^t \\ (r^\uparrow, t') & e \text{ is a leaf in } S^t,\ [e] = [t'] = [r], \text{ and } r \text{ the root of } S^{t'} \\ \text{undefined} & \text{otherwise} \end{cases}$$

where for a tuple $\bar{e} = e_1 \dots e_n$, we let $(\bar{e}, t)$ denote $(e_1, t) \dots (e_n, t)$.

Now, having determined the sets of edges and relations, it suffices to prove that this broad poset actually is a tree. We first note that any broad relation $u_{1,t_1} \dots u_{n,t_n} \leq t_v$ in $T \wedge (S^t)$ must satisfy the following properties:

(1) $t_i \leq_d t$ in T

(2) if $t_i \neq t$ then $t_i \leq_d t'$ for some $t' \in t^\uparrow$ such that $[r_{t'}] = [l_t]$ and $l \geq_d e$ in S^t

(3) if $t_i = t$, then $e_i \leq_d e$ in S^t

(4) if $t_i = t$ and $e_i = e$, then $n = 1$ and the relation is reflexive.

Indeed, these hold for the generating relation, and are preserved by broad transitivity.

Now, (1), (3), and (4) imply that the generated broad relation is antisymmetric, and hence we have a broad poset. Moreover, the dominant relation $\leq_d$ is also antisymmetric, and hence is itself an order relation.

Since it is clear that r_r, with r the root of S^r, is the only minimal element, it suffices to show that this broad poset is simple. Given $\bar{e} = (e_1, t_1) \dots (e_n, t_n) \leq (e, t)$, we go by downward induction on $\leq_d$ of (e, t). Since (e, t) is only the start of one non-trivial generating relation, we know that this factors $\bar{e} \leq (e, t)^\uparrow \leq (e, t)$, with $(e, t)^\uparrow = \epsilon_1 \dots \epsilon_m$, $\bar{e} = \bar{e}_1 \dots \bar{e}_m$, and $\bar{e}_i \leq \epsilon_i$. Here, $\epsilon_i = (\epsilon_i, t')$ for some fixed t', with either $t = t'$ or $t' \in t^\uparrow$ By the induction hypothesis, it suffices to show that the $\bar{e}_i$ have no edges in common. However, as the ϵ_i are $\leq_d$-incomparable in $S^{t'}$, this follows from (2) and (3) above. Hence, $T \wedge (S^t)$ is in fact a tree.

Finally, since T and each S^v are planar, the resulting tree has a natural inherited planar structure. $\square$

This construction provides our second map $d_0 : \mathbb{T}_1 \to \mathbb{T}_0$, sending $(T, (S^v))$ to $T \wedge (S^v)$.

We highlight a particular observation:

Lemma 3.1.13. *There is a natural "re-planarization" isomorphism filling the following diagram:*

$$\begin{tikzcd}
\mathbb{T}_1 \arrow[r, "\mathbb{V}"] \arrow[d, "d_0"'] \arrow[drr, Rightarrow, "\Phi_p"] & \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[r, "\mathsf{F}_0 \wr \mathbb{V}"] & \mathsf{F}_0{}^{\wr 2}\Sigma \arrow[d, "\mathrm{coll}"] \\
\mathbb{T}_0 \arrow[rr, "\mathbb{V}"'] & & \mathsf{F}_0 \wr \Sigma
\end{tikzcd}$$

Proof. We have $V(T \wedge (S^v)) = \amalg_{V(T)} V(S^v)$, and Φ_p just enforces the planar structure for $T(S^v)$ onto the collated set. $\square$

We have two sections s_0 and s_{-1} of d_0, defined by

$$s_0(T) = (T, (T_v)), \text{ the "trivial assembly" data, and}$$

$$s_{-1}(T) = (val(T), T), \text{ the "co-trivial assembly" data.}$$

We observe that $d_1 s_{-1}$ is not the identity, but instead equals $C_{(-)} \circ val$. These again look like simplicial identities; indeed, iterating this construction will create a simplicial object in categories.

Definition 3.1.14. Inductively, suppose $\mathbb{T}_k$ has been defined for $k < n$. Define $\mathbb{T}_n$ be the following pullback category:

$$\begin{tikzcd} \mathbb{T}_n \arrow[r, "\mathbb{V}"] \arrow[d, "d_n"'] & \mathsf{F}_0 \wr \mathbb{T}_{n-1} \arrow[d, "\mathsf{F}_0 \wr d_{n-1}"] \\ \mathbb{T}_{n-1} \arrow[r, "\mathbb{V}"] & \mathsf{F}_0 \wr \mathbb{T}_{n-2} \end{tikzcd}$$

Abusing notation, let val denote the composite $\mathbb{T}_n \xrightarrow{d_1 \ldots d_{n-1} d_n} \mathbb{T}_0 \xrightarrow{val} \Sigma$ for any $n \geq 1$. Since pullback squares are preserved by "stacking", we have the following:

Lemma 3.1.15. *$\mathbb{T}_n$ is equivalently the pullback*

$$\begin{tikzcd} \mathbb{T}_n \arrow[r, "\mathbb{V}"] \arrow[d, "val"] & \mathsf{F}_0 \wr \mathbb{T}_{n-1} \arrow[d, "\mathsf{F}_0 \wr val"] \\ \mathbb{T}_0 \arrow[r, "\mathbb{V}"] & \mathsf{F}_0 \wr \Sigma \end{tikzcd}$$

□

Lemma 3.1.16 (Pereira)**.** *$\mathbb{T}_n$ is equivalent to the category of "n-fold strings of planar tall maps": objects are strings of composable morphisms $T_0 \to T_1 \to \ldots \to T_n$ of planar tall maps, and morphisms are sets of connecting isomorphisms*

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, "\simeq"] & T_1 \arrow[r] \arrow[d, "\simeq"] & \ldots \arrow[r] & T_n \arrow[d, "\simeq"] \\ T_0' \arrow[r] & T_1' \arrow[r] & \ldots \arrow[r] & T_n' \end{tikzcd}$$

Moreover, these $\mathbb{T}_n$ naturally form a simplicial object in categories, and the assembly maps discussed above are induced by the obvious simplicial maps d_0 and d_n.

Proof. For $n = 1$, we note that any assembly data $(T, (S^v))$ induces a natural planar tall map $T \to T \wedge (S^v)$. Conversely, given a planar tall map $T \to S$ and a vertex $e^{\uparrow} \leq e$, define $S^v = S_{f(e^{\uparrow} \leq e)}$. Then $(T, (S^v))$ is a well-defined assembly data. Moreover, it is clear that these two constructions are mutually inverse.

Inductively, objects in $\mathbb{T}_n$ are trees T where each vertex is equipped with a string of $n-1$ planar tall arrows starting with T_v:

$$T_v \to S^{v,1} \to S^{v,2} \to \ldots \to S^{v,n-1}.$$

Starting with $T_v \to S^{v,1}$, we convert the first step into a planar tall map $T \to T \wedge S^{v,1} = T_1$, but now the planar tall maps $S^{*,1} \to S^{*,2}$ become assembly data for T_1. Continuing in this manner yields the desired result □

Under these identifications, the d_i and s_i are precisely as they would be for the nerve of a category.

3.2 Free Operads

In this section, we provide a new description of the free operad monad $\mathbb{F}$ as a particular left Kan extension, allowing for more categorical exploitation.

We begin with some review:

Definition 3.2.1. The *free operad* functor is the left adjoint of the forgetful functor below.

$$\text{fgt} : \mathcal{V}\mathsf{Op}_{\{*\}} \leftrightarrow \mathcal{V}^{\Sigma^{op}} : \mathbb{F} \tag{3.1}$$

The above definition repackages an explicit description of the free operad monad originally constructed by Spitzweck [Spi01, Proposition 5], and reformulated by Berger-Moerdijk [BM03] after Getzler-Kapronov [GK95]; it can also be found in [BO15] and [MSS07].

Recall the *nerve evaluation* and *valence* functors from Sections 2.3 and 3.1. Now, let $\mathbb{T}_0(n)/\sim$ be the set of isomorphism classes $[T]$ of trees T with $val(T) = n$. The Appendix of [BM03] states the following (using the inductive description of N_X):

Lemma 3.2.2. *Suppose $X \in \mathcal{V}^{\Sigma^{op}}$ is a symmetric sequence. For each $n \in \mathbb{N}_0$, the n-th level of the free operad generated by X is given by*

$$\coprod_{[T]\in\mathbb{T}_0(n)/\sim} N_X(T) \otimes_{\mathrm{Aut}(T)} \Sigma_n.$$

□

We will be using the naturality of the left Kan extension many times in this chapter in order to build desired maps. In Appendix A.1, we show that Lan is natural on the category $\mathsf{WSpan}(\Sigma, \mathcal{V})$, defined as follows. An object is a span $(\mathcal{C}, X)$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "X"] \arrow[d] & \mathcal{V} \\ \Sigma^{op} & \end{tikzcd}$$

while a map $(\mathcal{C}', X') \to (\mathcal{C}, X)$ consists of a pair (F, Φ)

$$\begin{tikzcd} \mathcal{C}' \arrow[d, "F"'] \arrow[dr, bend left, "X'"] \arrow[dd, bend right=60] & \\ \mathcal{C} \arrow[r, "X"'] \arrow[d] \arrow[ur, phantom, "\Leftarrow\Phi"] & \mathcal{V} \\ \Sigma^{op} & \end{tikzcd}$$

such that the left triangle commutes and Φ is a natural transformation filling the right triangle.

We begin by defining an endofunctor on the category of sequences.

Definition 3.2.3. Define the endofunctor $\mathbb{F} : \mathcal{V}\mathsf{Sym} \to \mathcal{V}\mathsf{Sym}$ by sending a sequence X to the left Kan extension

$$\begin{tikzcd} \mathbb{T}_0^{op} \arrow[r, "N_X"] \arrow[d, "val"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{V} \\ \Sigma^{op} \arrow[ur, "\mathbb{F}X := \mathrm{Lan}_{val} X"'] & \end{tikzcd}$$

Since maps $X \to Y$ induce maps $N_X \Rightarrow N_Y$, functoriality on morphisms is given by applying

Lemma A.1.2 to the diagram

$$
\begin{tikzcd}
\mathbb{T}_0^{op} \arrow[r, "\mathbb{V}"] \arrow[d, "val"'] & (\mathsf{F}_0 \wr \Sigma)^{op} \arrow[r, bend left, "\mathsf{F}_0 \wr X"] \arrow[r, bend right, "\mathsf{F}_0 \wr Y"'] \arrow[r, phantom, "\Downarrow \mathsf{F}_0 \wr \varphi"] & (\mathsf{F}_0 \wr \mathcal{V}^{op})^{op} \arrow[r] & \mathcal{V} \\
\Sigma^{op} \arrow[urrr, dotted, bend right] \arrow[urrr, dotted, bend right=40] & & &
\end{tikzcd}
$$

The resulting arrow is characterized as the unique map such that the following diagram commutes.

$$
\begin{tikzcd}
N_X \arrow[r, Rightarrow, "\mathsf{F}_0 \wr \varphi"] \arrow[d, Rightarrow, "\alpha_X"'] & N_Y \arrow[d, Rightarrow, "\alpha_Y"] \\
\mathbb{F}X \circ val \arrow[r, Rightarrow, "L(\mathsf{F}_0 \wr \varphi) \circ val"] & \mathbb{F}Y \circ val
\end{tikzcd}
$$

Unpacking definitions, we immediately have the following:

Lemma 3.2.4. $\mathbb{F}(X)$ *is levelwise isomorphic to the description of the free operad on X given by Lemma 3.2.2.* □

We will now build a monad structure onto this endofunctor.

Remark 3.2.5. In many of the larger diagrams in the remainder of this section, we will be drawing the opposite of the diagrams we desire. This is strictly for notational convenience: the prevalence of the notation $(-)^{op}$ becomes detrimental very quickly to visually following the flow of information, and drawing the opposite diagram minimizes their appearances.

Using the categories $\mathbb{T}_n$ of "iterated assembly data", we can iterate $\mathbb{F}$:

Lemma 3.2.6. *Given a sequence X, the sequence $\mathbb{F}\mathbb{F}X$ is isomorphic to the left Kan extension*

$$\mathbb{F}\mathbb{F}X \simeq \mathrm{Lan}_{val \circ \pi_0}(\otimes \circ \mathsf{F}_0 \wr N_X \circ \mathbb{V}).$$

Proof. Denote the given left Kan extension L. Consider the (opposite of the) following commuting diagram (where empty 2-cells are the identity), whose outermost span is the defining span for L:

$$
\begin{tikzcd}
\mathbb{T}_1 \arrow[r, "\mathbb{V}"] \arrow[d, "d_1"'] & \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[rr, "\mathsf{F}_0 \wr N_X"] \arrow[d, "\mathsf{F}_0 \wr val"'] \arrow[drr, phantom, "\Downarrow \mathsf{F}_0 \wr \alpha_{\mathbb{F}X}"] & & \mathsf{F}_0 \wr \mathcal{V} \arrow[r, "\otimes"] & \mathcal{V} \\
\mathbb{T}_0 \arrow[r, "\mathbb{V}"] \arrow[d, "val"] & \mathsf{F}_0 \wr \Sigma \arrow[urr, "\mathsf{F}_0 \wr \mathbb{F}X"'] \arrow[urrr, bend right, "L'"'] & & & \\
\Sigma \arrow[uurrrr, dotted, bend right, "L"'] & & & &
\end{tikzcd}
\tag{3.2}
$$

By Lemma A.1.8, L' is the left Kan extension of $\otimes \circ \mathsf{F}_0 \wr N_X$ over $\mathsf{F}_0 \wr val$. But then Lemma A.1.9 implies that $L'\mathbb{V}$ is the left Kan extension of the top row. However, we note that $L'\mathbb{V} = N_{\mathbb{F}X}$ is the nerve evaluation map for $\mathbb{F}X$, whose left Kan extension over val is precisely $\mathbb{F}\mathbb{F}X$. Thus, the result is proved by applying Lemma A.0.2. $\square$

We denote the solid natural transformation in Diagram (3.2) by d_1'.

Now, for any sequence X, the above plus Lemma A.1.2 provides the description of a natural map $\mu_X : \mathbb{F}\mathbb{F}X \to \mathbb{F}X$, via the (opposite of the) following diagram in $\mathsf{WSpan}(\Sigma, \mathcal{V})$; we denote the rectangular natural transformation by d_0'.

$$
\begin{tikzcd}
\mathbb{T}_1 \arrow[r, "\mathbb{V}"] \arrow[d, "d_0"'] & \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[r, "\mathsf{F}_0 \wr \mathbb{V}"] \arrow[d, phantom, "\Updownarrow \Phi_p"] & \mathsf{F}_0{}^{\wr 2}\Sigma \arrow[r, "\mathsf{F}_0{}^{\wr 2} X"] \arrow[d, "\mathrm{coll}"] & \mathsf{F}_0{}^{\wr 2}\mathcal{V}^{op} \arrow[r, "\mathsf{F}_0 \wr \otimes"] \arrow[d, "\mathrm{coll}"] & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r, "\otimes"] \arrow[d, phantom, "\Phi_\otimes \Updownarrow"] & \mathcal{V}^{op} \arrow[d, equal] \\
\mathbb{T}_0 \arrow[rr, "\mathbb{V}"'] \arrow[d, "val"'] & & \mathsf{F}_0 \wr \Sigma \arrow[r, "\mathsf{F}_0 \wr X"] & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[rr, "\otimes"'] & & \mathcal{V}^{op} \\
\Sigma & & & & &
\end{tikzcd}
\tag{3.3}
$$

We also have a natural map $\epsilon : X \to \mathbb{F}X$ induced similarly; denote this natural transformation s':

$$
\begin{tikzcd}
\Sigma \arrow[rrr, "X"] \arrow[d, "s=C_{(-)}"'] \arrow[dr, "\sigma"] & & & \mathcal{V}^{op} \arrow[d, equal] \\
\mathbb{T}_0 \arrow[r, "\mathbb{V}"] & \mathsf{F}_0 \wr \Sigma \arrow[r] & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op}
\end{tikzcd}
\tag{3.4}
$$

where $C_{(-)}$ is the inclusion of the corollas.

Theorem 3.2.7. *The triple $(\mathbb{F}, \mu, \epsilon)$ is a monad on the category of symmetric sequences.*

We delay the proof until Chapter 6, as this is the special case $G = \{*\}$ of Theorem 6.3.6.

Corollary 3.2.8. $\mathbb{F}$ *is the free operad monad.* $\square$

This allows us to repackage much of the additional data of an operad. In particular, an operad $\mathcal{P}$ comes equipped with a natural transformation $\tilde{\mu} : N_{\mathcal{P}} \Rightarrow \mathcal{P} \circ val$,

$$\begin{tikzcd}
\mathbb{T}_0^{op} \arrow[r, "N_{\mathcal{P}}"] \arrow[d, "val"'] \arrow[dr, phantom, "\Searrow"] & \mathcal{V} \\
\Sigma^{op} \arrow[ur, "\mathcal{P}"'] &
\end{tikzcd}$$

given by $\mu \circ \alpha$, where $\mu : \mathbb{F}\mathcal{P} \Rightarrow \mathcal{P}$ and $\alpha_{\mathcal{P}} : N_{\mathcal{P}} \Rightarrow \mathbb{F}\mathcal{P} \circ val$. All together, this can be packaged as follows:

Lemma 3.2.9. *An $\mathbb{F}$-algebra structure on X is equivalent to the data of a morphism $\tilde{\mu}$: $N_X \Rightarrow X \circ val$ such that*

(1) (unitality) $\tilde{\mu}$ is the identity on corollas; and

(2) (associativity) the following two (compositions of) natural transformations are equal:

$$\begin{tikzcd}
\mathbb{T}_1^{op} \arrow[r, "\mathbb{V}"] \arrow[d, "d_1"'] & \mathsf{F} \wr \mathbb{T}_0^{op} \arrow[r, "\mathsf{F}\wr N_X"] \arrow[d, "val"'] \arrow[dr, phantom, "\Searrow \mathsf{F}\wr\tilde{\mu}"] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\otimes"] & \mathcal{V} \arrow[d, equal] \\
\mathbb{T}_0^{op} \arrow[r, "\mathbb{V}"] \arrow[d, "val"'] & \mathsf{F} \wr \Sigma^{op} \arrow[rr] \arrow[ur, bend right, "\mathsf{F}\wr X"'] \arrow[d, phantom, "\Downarrow \tilde{\mu}"] & & \mathcal{V} \arrow[d, equal] \\
\Sigma^{op} \arrow[rrr, "X"'] & & & \mathcal{V}
\end{tikzcd} \tag{3.5}$$

and

$$\begin{tikzcd}
\mathbb{T}_1^{op} \arrow[r, "\mathbb{V}"] \arrow[d, "d_0"'] & \mathsf{F} \wr \mathbb{T}_0^{op} \arrow[r, "\mathsf{F}\wr\mathbb{V}"] \arrow[rrrr, bend left, "\mathsf{F}\wr N_X"] \arrow[d, phantom, "\Updownarrow \Phi_p"] & \mathsf{F} \wr \mathsf{F} \wr \Sigma^{op} \arrow[r, "\mathsf{F}\wr\mathsf{F}\wr X"] \arrow[d, "\mathrm{coll}"'] & \mathsf{F} \wr \mathsf{F} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\otimes"] \arrow[d, "\mathrm{coll}"'] \arrow[dr, phantom, "\Updownarrow \Phi_{\otimes}"] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\otimes"] & \mathcal{V} \arrow[d, equal] \\
\mathbb{T}_0^{op} \arrow[rr, "\mathbb{V}"'] \arrow[d, "val"'] & & \mathsf{F} \wr \Sigma^{op} \arrow[r, "\mathsf{F}\wr X"'] \arrow[d, phantom, "\Downarrow \tilde{\mu}"] & \mathsf{F} \wr \mathcal{V} \arrow[rr, "\otimes"'] & & \mathcal{V} \arrow[d, equal] \\
\Sigma \arrow[rrrrr, "X"] & & & & & \mathcal{V}
\end{tikzcd} \tag{3.6}$$

where $\Phi_\otimes$ and Φ_p are the re-collating and re-planarizing natural isomorphisms.

□

Explicitly, a symmetric sequence X is an operad if and only if, for all trees T, we have structure maps $N_X(T) \to X(val(T))$ which are associative, unital, and $\mathrm{Aut}(T)$-equivariant.

Remark 3.2.10. We unpack multiplicative unitality, as it is recorded as a synthesis of the conditions given in the previous lemma. The key observation is that $val(\eta) = C_1$, but $N_X(\eta) = I$ is the monoidal unit. Now, given a tree T, let T_+ denote $T \circ_1 C_1$, and note that $val(T) = val(T_+)$. We consider the object $(T_+, (S^v))$ in $\mathbb{T}_1$ where $S^v = T_v$ if $v \in V(T)$, and $S^v = \eta$ if v is the added node $+$. Now, the compatibilities of Lemma 3.2.9 say precisely that the two maps

$$N_X(T) \otimes I \xrightarrow{id \otimes \tilde{\mu}(\eta)} N_X(T) \otimes X(1) \xrightarrow{\tilde{\mu}} X(val(T))$$

$$N_X(T) \otimes * =\!=\!=\!=\!=\!= N_X(T) \xrightarrow{\tilde{\mu}} X(val(T))$$

are equal.

We can use this framework to build the canonical split coequalizer for an algebra over a monad:

$$\mathbb{F}\mathbb{F}X \underset{\alpha_{\mathbb{F}X}}{\overset{\mathbb{F}\mu}{\rightrightarrows}} \mathbb{F}X \xrightarrow{\mu} X$$

We recall that a split coequalizer in $\mathcal{C}$ consists of the following data:

$$\begin{tikzcd} A \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & B \arrow[l, "s_0"'] \arrow[l, bend right, "s_{-1}"'] \arrow[r, "d"] & C \arrow[l, bend right, "s"'] \end{tikzcd} \tag{3.7}$$

such that s_0 is a section of both d_i, s is a section of d, and $sd = s_{-1}d_1$.

Lemma 3.2.11. *The maps*

$$\begin{tikzcd}
\mathbb{T}_1^{op} \arrow[r, "N_X^1"] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_0"] & \mathcal{V} \\
\mathbb{T}_0^{op} \arrow[ur, "N_X^0"'] \arrow[d, "val"'] & \\
\Sigma^{op} \arrow[uur, bend right, "X"'] &
\end{tikzcd}$$

are part of a split coequalizer diagram in $\mathsf{WSpan}(\Sigma, \mathcal{V})$.

Proof. We recall that split coequalizers are *absolute*: any diagram of the form (3.7) must have d a coequalizer of the pair (d_0, d_1). Thus, it will suffice for us to just build maps satisfying the above relations. We define:

$$d_0 = (d_0, d_0')$$
$$d_1 = (d_1, d_1')$$
$$d = (val, \tilde{\mu}).$$

We note that the natural transformation d is precisely the bottom portion of both Diagrams 3.6 and 3.5, and hence Lemma 3.2.9 implies that it equalizes d_0 and d_1.

For the degeneracies:

$$s_0 = (s_0, id)$$
$$s_{-1} = (s_{-1}, id)$$
$$s = (C_{(-0}, id).$$

We observed previously that s_0 and s_{-1} are sections of d_0, and that s_{-1} is a section of d_0.

It remains to verify that $d_1 s_{-1} = sd$. On functors, this is clear, as both send T to $C_{val(T)}$. On natural transformations, this is straightforward, as both are just $\tilde{\mu}$; the $\mathsf{F}_0 \wr (-)$ in $d_1 s_{-1}$ is recording an indexing over a singleton set, and hence has no effect.

Thus we have an absolute coequalizer diagram, and hence a (split) coequalizer in

$\mathsf{WSpan}(\Sigma, \mathcal{V})$. $\square$

This gives us our categorical description of the canonical coequalizer:

Corollary 3.2.12. *X is the coequalizer of the induced maps $d_0, d_1 : \mathbb{F}\mathbb{F}X \rightrightarrows \mathbb{F}X$.*

Proof. This follows from Lemma 3.2.11, and the identifications from Lemma 3.2.4, Lemma 3.2.6, and Theorem 3.2.7, by an application of Proposition A.1.2. $\square$

We can also repackage the property of being a morphism of operads using this formalism. Recall that a map of sequences $\varphi : X \to Y$ between $\mathbb{F}$-algebras is an $\mathbb{F}$-algebra *morphism* if the diagram

$$\begin{tikzcd}
\mathbb{F}X \arrow[r, "\mathbb{F}\varphi"] \arrow[d, "\alpha_X"'] & \mathbb{F}Y \arrow[d, "\alpha_Y"] \\
X \arrow[r, "\Phi"] & Y
\end{tikzcd}$$

commutes. Combining this with the universal property of $\mathbb{F}\varphi$ from above, we see that the following two diagrams of natural transformations are equal:

$$\begin{tikzcd}
\mathbb{T}_0 \arrow[r] \arrow[d] & \mathsf{F}_0 \wr \Sigma \arrow[rr, bend left, "\mathsf{F}_0 \wr X"] \arrow[rr, bend right, "\mathsf{F}_0 \wr Y"'] \arrow[rr, phantom, "\Downarrow \mathsf{F}_0 \wr \varphi"] & & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r] \arrow[d, phantom, "\Downarrow \alpha_Y"] & \mathcal{V}^{op} \\
\Sigma \arrow[rrrru, bend right=15, "\mathbb{F}Y"] \arrow[rrrru, bend right=50, "Y"'] \arrow[rrr, phantom, "\Downarrow \mu_Y"] & & & &
\end{tikzcd} \tag{3.8}$$

$$\begin{tikzcd}
\mathbb{T}_0 \arrow[r] \arrow[d] \arrow[rr, phantom, "\Downarrow \alpha_X"'] & \mathsf{F}_0 \wr \Sigma \arrow[rr, "\mathsf{F}_0 \wr X"] & & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \\
\Sigma \arrow[rrrru, "\mathbb{F}X"] \arrow[rrrru, bend right=15, "\mathbb{F}Y"'] \arrow[rrrru, bend right=50, "Y"'] \arrow[rrr, phantom, "\Downarrow \mathbb{F}\varphi"] \arrow[rrr, phantom, "\Downarrow \mu_Y"'] & & & &
\end{tikzcd} \tag{3.9}$$

Lemma 3.2.13. *A map of sequences $\varphi : X \to Y$ is an $\mathbb{F}$-algebra morphism* IFF *the two diagrams of natural transformations below agree:*

$$
\begin{tikzcd}
\mathbb{T}_0 \arrow[r] \arrow[d] & \mathsf{F}_0 \wr \Sigma \arrow[rr, bend left, "\mathsf{F}_0 \wr X"] \arrow[rr, bend right, "\mathsf{F}_0 \wr Y"'] \arrow[rr, phantom, "\Downarrow \mathsf{F}_0 \wr \varphi"] & & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r] \arrow[d, phantom, "\Downarrow \tilde{\mu}_Y"] & \mathcal{V}^{op} \\
\Sigma \arrow[rrrru, bend right, "Y"'] & & & &
\end{tikzcd}
\tag{3.10}
$$

$$
\begin{tikzcd}
\mathbb{T}_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \tilde{\mu}_X"] & \mathsf{F}_0 \wr \Sigma \arrow[rr, "\mathsf{F}_0 \wr X"] & & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \\
\Sigma \arrow[rrrru, "X"] \arrow[rrrru, bend right, "Y"'] \arrow[rrrru, phantom, "\Downarrow \varphi"'] & & & &
\end{tikzcd}
\tag{3.11}
$$

Proof. We first note that Diagram (3.10) is equal to Diagram (3.8) by definition of $\tilde{\mu}$. Now, the "only if" is immediate from the discussion before the lemma and the universal property of $\mathbb{F}\varphi$. The "if" follows by the universal property of coproducts and the explicit description of $\mathbb{F}X(n)$ and α_Y, and the observation that *val* is (split) surjective as an arrow in the category of categories, after again noting $\tilde{\mu}_X = \mu_X \circ \alpha_X$. $\square$

Remark 3.2.14. Using the above, it is easy to check that the maps induced by Lemma 3.2.11 are indeed operad maps.

3.3 The Coproduct of Operads

In the previous section, we built the operads $\mathbb{F}Y$ and $\mathbb{F}\mathbb{F}Y$ as left Kan extensions over categories of trees. In this subsection, we expand the previous description to build the operadic coproduct $\amalg \mathbb{F}Y_i \amalg \mathcal{P}$ for arbitrary sequences $Y_1, \ldots, Y_{n-1}$ and operads $\mathcal{P}$ as a left Kan extension (Proposition 3.3.6).

3.3.1 Passive Labeled Trees

We begin this process by constructing appropriate categories of trees, whose nodes are *labeled* as either "active" of "passive" (with multiple possible flavors of passive labelings). Eventually, these active nodes will correspond to the operad $\mathcal{P}$, the passive nodes to the various Y_i, and the flexibility on the active nodes to the composition structure maps in $\mathcal{P}$.

The choice we make for our fundamental category of labeled trees is not necessarily where one would expect to start, as it is *not* "trees with labelings". Instead, our base category is more rigid, which will eventually provide us with a cleaner presentation.

We first make a quick definition:

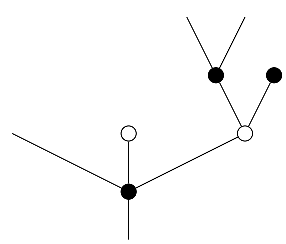
Definition 3.3.1. Given a tree T, the *height* of a vertex is the number of vertices (inclusive) in the unique descending path from that vertex to the root vertex. We call a vertex *odd* if the height is odd. We call a tree *odd* if $V(T) \neq \varnothing$ and every leaf vertex is odd.

Equivalently, a tree is odd if every leaf-root path has an odd number of vertices in it.

Example 3.3.2. Some examples:

(1) the root vertex is always odd, and corollas are always odd trees;

(2) the black nodes below are all odd, as is the tree itself: the stumps are not part of any leaf-root path, and hence in particular the white stump does not affect the parity of the tree.



Definition 3.3.3. Let $\lambda_1^n\mathbb{T}_{-1}$ be the category of "alternating n-passive trees" and isomorphisms, described as follows. Objects are odd (planar) trees T equipped with a labeling

$\lambda : V(T) \to \{a, p_1, \dots, p_n\}$ such that a vertex is in the preimage of a IFF it is odd. Morphisms $(T, \lambda) \to (T', \lambda')$ are non-planar isomorphisms of the underlying trees $\varphi : T \to T'$ which preserve labelings.

Notationally, we call vertices in $\lambda^{-1}(a)$ *active*, and denote the set of active nodes $V_a(T)$; similarly, the preimage $\lambda^{-1}(\{p_1, \dots, p_n\})$ will be denoted $V_p(T) = \amalg V_{p_i}(T)$ as the set of *passive nodes*, with $V_{p_i}(T)$ the set of p_i-labeled passive nodes.

More explicitly, alternating n-passive trees are odd trees such that the root and leaf vertices are active, as is every alternating vertex coming up from the root. The remaining nodes may be tagged by any of the passive labelings. For this reason, these may also be called "n-alternating trees".

These alternating n-passive trees form the foundation of our study of the desired coproduct. The "-1" notation is suggestive, and will become more natural as we move through the section.

We now generalize the definitions from Section 3.1 into the alternating setting.

Definition 3.3.4. Define the alternating n-passive *vertex functor*

$$\mathbb{V} = (\mathbb{V}_{p_i})^{\times n} \times \mathbb{V}_a : \lambda_1^n \mathbb{T}_{-1} \to (\mathsf{F_0} \wr \Sigma)^{\times n} \times \mathsf{F_0} \wr \Sigma$$

as the mapping

$$(T, \lambda) \mapsto (V_{p_i}(T), v \mapsto T_v)^{\times n} \times (V_a(T), v \mapsto T_v),$$

partitioning $V(T)$ into its labeled pieces.

Definition 3.3.5. Given sequences Y_i and X, define the alternating n-passive *nerve evaluation* functor $N^{-1}_{(Y_i),X} : \lambda_1^n \mathbb{T}_{-1}^{op} \to \mathcal{V}^{op}$ to be the opposite of the composite

$$\lambda_1^n \mathbb{T}_{-1} \xrightarrow{\mathbb{V}} (\mathsf{F_0} \wr \Sigma)^{\times n} \times \mathsf{F_0} \wr \Sigma \xrightarrow{(\mathsf{F_0} \wr Y_i)^{\times n} \times \mathsf{F_0} \wr X} (\mathsf{F_0} \wr \mathcal{V}^{op})^{\times n} \times \mathsf{F_0} \wr \mathcal{V}^{op} \xrightarrow{\otimes} \mathcal{V}^{op} \tag{3.12}$$

where the last arrow is $(n+1)$-copies of $\otimes$ followed by $\otimes : \mathcal{V}^{n+1} \to \mathcal{V}$.

If the sequences Y_i and X have been fixed, we will denote this functor just by N. Explicitly, N sends an alternating n-passive tree T to the tensor product

$$\bigotimes_i \bigotimes_{v \in V_{p_i}(T)} Y_i(T_v) \otimes \bigotimes_{v \in V_a(T)} X(T_v).$$

There is also natural map $\mathsf{fl} : \lambda_1^n \mathbb{T}_{-1} \to \mathbb{T}_0$ which just forgets labels. Abusing notation as before, we call the composite $val \circ \mathsf{fl}$ also by just val.

We will spend the remainder of this section proving the following proposition:

Proposition 3.3.6. *The sequence underlying the operad $\amalg_i \mathbb{F} Y_i \amalg \mathcal{P}$ is equivalent to the left Kan extension*

$$\mathrm{Lan}_{val} \, N^{-1}_{(Y_i),\mathcal{P}}.$$

3.3.2 The Coproduct of Free Operads

We begin with an easier construction, namely $\amalg \mathbb{F} Y_i \amalg \mathbb{F} X$ for generic sequences Y_i and X.

Remark 3.3.7. The asymmetry will not be important for this section, but provides the framework for replacing $\mathbb{F}X$ with $\mathcal{P}$.

Again, this process will be rooted in constructing appropriate categories of trees.

Definition 3.3.8. Define $\lambda_1^n \mathbb{T}_0$ to be the category of "alternating n-passive assembly data" and isomorphisms, given by the pullback

$$\begin{tikzcd}
\lambda_1^n \mathbb{T}_0 \arrow[r, "\mathbb{V}"] \arrow[d, "d"'] & (\mathsf{F}_0 \wr \Sigma)^{\times n} \times \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[d, "id \times \mathsf{F}_0 \wr val"] \\
\lambda_1^n \mathbb{T}_{-1} \arrow[r, "\mathbb{V}"] & (\mathsf{F}_0 \wr \Sigma)^{\times n} \times \mathsf{F}_0 \wr \Sigma
\end{tikzcd}$$

Explicitly, objects are alternating trees T such that each active node T_v is equipped with a planar tall map $T_v \to S_v$. We will refer to the top map $\mathbb{V}$ as the "n-passive *vertex functor*".

We have an obvious section $s : \lambda_1^n\mathbb{T}_{-1} \to \lambda_1^n\mathbb{T}_0$ sending T to the trivial active assembly data $(T, (T_v))$.

This construction is mimeographing a more obvious concept, but provides the asymmetric framing that we will need.

Definition 3.3.9. Define the category $\lambda^n\mathbb{T}_0$ of "n-labeled trees" and isomorphisms to be the category with objects trees T equipped with a labeling $\lambda : V(T) \to \{a, p_1, \dots, p_n\}$, with label-preserving tree isomorphisms.

Lemma 3.3.10 (Pereira). *$\lambda_1^n\mathbb{T}_0$ is isomorphic to $\lambda^{n+1}\mathbb{T}_0$.*

Proof. The assembly functor $d_0 : \mathbb{T}_1 \to \mathbb{T}_0$ extends naturally to a functor $d_0 : \lambda_1^n\mathbb{T}_0 \to \lambda^{n+1}\mathbb{T}$, where all "new" vertices are labeled as active. Conversely, given an $(n+1)$-labeled tree T, we will define an n-alternating tree T^{alt} and associated assembly data. Let $\left\{S^t\right\}$ denote the collection of maximal subtrees of T with only active vertices, with t the root edge of S^t; we particularly include subtrees with *zero* active nodes, corresponding to an edge adjacent to two passive vertices. Heuristically, T^{alt} will be the broad poset with relations generated by the $val(S^t)$ and the passive vertices.

Explicitly, we first consider the case where none of the S^t are sticks; that is, T has no adjacent passive nodes. We define T^{alt} to be the inner face of T obtained by removing all inner edges of the S^t. We note that this is indeed alternating, and moreover $(T, (S^t)$ is alternating assembly data for T.

If some S^t are sticks, we produce T^{alt} in two steps:

$$T^{alt} \to T_{/a} \to T$$

First, build $T_{/a}$ from T as above, but this time ignoring those S^t which are sticks. $T_{/a} \in \lambda^{n+1}\mathbb{T}_0$ will *not* be alternating, but will not have any adjacent active nodes. Next, construct T^{alt} from $T_{/a}$ by adding a new edge e_t' for each $S^t = \{e_t\}$ which is a stick, and two new generating relations: $(e_t') \le e_t$ and $(e_t)^{\uparrow} \le e_t'$ (that is, split the edge of S^t by adding a new unary vertex). Now, there is a natural degeneracy map $T^{alt} \to T_{/a}$ which sends both e_t and e'_t to e_t. Moreover, T^{alt} is now alternating, and $(T^{alt}, (S^t))$ is again alternating assembly data for T.

Finally, there is always a map $T \to d_0(T, (S^v)) = T \wedge (S^v)$ in $\lambda^{n+1}\mathbb{T}_0$ whose underlying map of trees is planar tall. If we consider the subset $\{S^w\} \subseteq \{S^v\}$ of those selected maximal subtrees which are sticks, then

$$T \to T \wedge (S^w) \to T \wedge (S^v)$$

is an (inner)face-degeneracy decomposition of this map, with $T \wedge (S^w)$ precisely $(T \wedge (S^v))_{/a}$. From this observation, plus the uniqueness of these decompositions of planar tall maps, we conclude that these functors defined above are mutually inverse. □

Diagrammatically, we have the following:

$$\begin{tikzcd} \lambda_1^n\mathbb{T}_0 \arrow[rr, shift left, "d_0"] \arrow[rr, phantom, "\simeq"] \arrow[dr, "d"'] & & \lambda^{n+1}\mathbb{T}_0 \arrow[ll, shift left, "(-)^{alt}"] \arrow[dl, "(-)^{alt}"] \\ & \lambda_1^n\mathbb{T}_{-1} & \end{tikzcd}$$

We have two natural sections of d, inspired by the non-labeled case, namely $s_0 : T \mapsto (T, (T_v))$ and $s_{-1} : T \mapsto (T^{alt}, (S^t))$, with the S^t defined as in the proof above.

We now lift the definitions from the previous section:

Definition 3.3.11. Given sequences Y_i and X, define the n-passive *nerve evaluation* functor $N^0_{(Y_i),X} : \lambda_1^n\mathbb{T}_0^{op} \to \mathcal{V}$ to be the opposite of the composite

$$\lambda_1^n\mathbb{T}_0 \xrightarrow{\mathbb{V}} \mathsf{F}_0 \wr \mathbb{T}_0 \xrightarrow{\mathsf{F}_0 \wr \mathbb{V}} \mathsf{F}_0 \wr \mathsf{F}_0 \wr \Sigma \xrightarrow{\mathsf{F}_0 \wr \mathsf{F}_0 \wr X} \mathsf{F}_0 \wr \mathsf{F}_0 \wr \mathcal{V}^{op} \xrightarrow{\mathsf{F}_0 \wr \otimes} \mathsf{F}_0 \wr \mathcal{V}^{op} \xrightarrow{\otimes} \mathcal{V}^{op}$$

where we have omitted what happens for passive nodes, as it is identical to Equation (3.12).

Remark 3.3.12. It is clear that $N^{-1}_{(Y_i),X}$ is equal to the composite $N^0_{(Y_i),X} \circ s$.

Abusing notation, denote also by d_0 the composite

$$\lambda_1^n \mathbb{T}_0 \xrightarrow{d_0} \lambda_1^n \tilde{\mathbb{T}}_0 \xrightarrow{\mathsf{fl}} \mathbb{T}_0.$$

We now come to the main result of this subsection:

Lemma 3.3.13. *The coproduct in operads* $\amalg \mathbb{F} Y_i \amalg \mathbb{F} X$ *is isomorphic to the left Kan extension*

$$\begin{tikzcd}
\lambda_1^n \mathbb{T}_0^{op} \arrow[r, "N^0_{(Y_i),X}"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\
\mathbb{T}_0^{op} \arrow[d, "val"'] & \\
\Sigma^{op} \arrow[uur, "\operatorname{Lan}_{val} N^0_{(Y_i),X}"'] &
\end{tikzcd}$$

Proof. By Lemma A.0.2, it suffices to show that $N_{\amalg Y_i \amalg X} = \operatorname{Lan}_{d_0} N^0_{(Y_i),X}$;

$$\begin{tikzcd}
\lambda_1^n \mathbb{T}_0^{op} \arrow[r, "N^0_{(Y_i),X}"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\
\mathbb{T}_0^{op} \arrow[ur, "N_{\amalg Y_i \amalg X}"'] \arrow[d, "val"'] & \\
\Sigma^{op} \arrow[uur, dotted, bend right, "\operatorname{Lan}_{val \circ \mathsf{fc}} N \simeq \operatorname{Lan}_{val} N"'] &
\end{tikzcd}$$

This is clear on objects:

$$\begin{aligned}
\operatorname{Lan}_{\mathsf{fl}} N^0_{(Y_i),X}(T) &\simeq \operatorname*{colim}_{T \downarrow \mathsf{fl}(T',\lambda')} N^0_{(Y_i),X}(\mathsf{fl}(T',\lambda)) \simeq \operatorname*{colim}_{T \simeq \mathsf{fl}(T',\lambda')} N^0_{(Y_i),X}(T) \\
&\simeq \coprod_{\frac{\lambda : V(T) \to \{a, p_1, \ldots, p_n, b\}}{\operatorname{Aut}(T)}} \left(N^0_{(Y_i),X}(T,\lambda) \bigotimes_{\operatorname{Aut}(T,\lambda)} \operatorname{Aut}(T) \right) \\
&\simeq \coprod_{\frac{\lambda : V(T) \to \{a, p_1, \ldots, p_n, b\}}{\operatorname{Aut}(T)}} \bigotimes_{v \in V(T)} Y_{\lambda(v)}(T_v) \bigotimes_{\operatorname{Aut}(T,\lambda)} \operatorname{Aut}(T)
\end{aligned}$$

$$\simeq N_{\amalg Y_i \amalg X}(T).$$

where we have denoted $Y_0 = X$ for convenience. Lastly, since fl is full and morphisms just act by permuting indices, it is clear that $N_{\amalg Y_i \amalg X}$ is in fact the left Kan extension on morphisms as well. $\square$

For our next step, we will now categorically describe the free operad $\mathbb{F}(\amalg Y_i \amalg \mathbb{F}X)$. Again, we begin by defining a new layer in our tower of alternating trees.

Definition 3.3.14. Let $\lambda_1^n \mathbb{T}_1$ be the category of "alternating n-passive iterated assembly data", given by the pullback

$$\begin{tikzcd}
\lambda_1^n \mathbb{T}_1 \arrow[r, "\mathbb{V}_p \times \mathbb{V}_a"] \arrow[d, "d_1"] & (\mathsf{F}_0 \wr \Sigma)^{\times n} \times \mathsf{F}_0 \wr \mathbb{T}_1 \arrow[d, "id \times \mathsf{F}_0 \wr d_1"] \\
\lambda_0^n \mathbb{T}_0 \arrow[r, "\mathbb{V}_p \times \mathbb{V}_a"] & (\mathsf{F}_0 \wr \Sigma)^{\times n} \times \mathsf{F}_0 \wr \mathbb{T}_0
\end{tikzcd}$$

Explicitly, we have that objects consist of a pair $(T, (S^v \to W^v))$ with

(1) and alternating n-passive tree T, and

(2) planar tall maps $T_v \to S^v \to W^v$ for each $v \in V_a(T)$.

Now, consider the (opposite of the) following diagram, where we have omitted the parts referring to the passive nodes:

$$\begin{tikzcd}
\lambda_1^n \mathbb{T}_1 \arrow[r, "\mathbb{V}_a"] \arrow[d, "d_1"] & \mathsf{F}_0 \wr \mathbb{T}_1 \arrow[r, "\mathsf{F}_0 \wr \mathbb{V}"] \arrow[d, "\mathsf{F}_0 \wr d_1"] & \mathsf{F}_0^{\wr 2} \mathbb{T}_0 \arrow[r, "\mathsf{F}_0^{\wr 2} \mathbb{V}"] \arrow[d, "\mathsf{F}_0^{\wr 2} val"] \arrow[dr, Rightarrow] & \mathsf{F}_0^{\wr 3} \Sigma \arrow[r, "\mathsf{F}_0^{\wr 2} X^{\otimes}"] & \mathsf{F}_0^{\wr 2} \mathcal{V}^{op} \arrow[r, "\mathsf{F}_0 \wr \otimes"] & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r, "\otimes"] & \mathcal{V}^{op} \\
\lambda_1^n \mathbb{T}_0 \arrow[r, "\mathbb{V}_a"] & \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[r, "\mathsf{F}_0 \wr \mathbb{V}"] & \mathsf{F}_0^{\wr 2} \Sigma \arrow[urr, "\mathsf{F}_0^{\wr 2} \mathbb{F}X"'] \arrow[urrr, bend right, "\mathsf{F}_0 \wr L_1"'] \arrow[urrrr, bend right, dashed, "\mathsf{L}_2"'] & & & &
\end{tikzcd} \tag{3.13}$$

Here, L_1 is the (opposite of the) composite

$$L_1^{op} : \mathsf{F}_0 \wr \Sigma \xrightarrow{\mathsf{F}_0 \wr \mathbb{F}X} \mathsf{F}_0 \wr \mathcal{V}^{op} \xrightarrow{\otimes} \mathcal{V}^{op}$$

while L_2 is a composite which includes the information omitted in the above diagram;

specifically, it is the (opposite of the) composite

$$L_2^{op} : (\mathsf{F}_0 \wr \Sigma)^{\times n} \times \mathsf{F}_0 \wr (\mathsf{F}_0 \wr \Sigma) \xrightarrow{(Y_i)^{\otimes} \times \mathsf{F}_0 \wr (L)} \mathcal{V}^{op} \times \mathsf{F}_0 \wr (\mathcal{V}^{op}) \xrightarrow{\otimes} \mathcal{V}^{op}$$

where $(Y_i)^{\otimes}$ is a synthesis of the individual $Y_i^{\otimes}$, given as the (opposite of the) composite

$$(Y_i^{\otimes})^{op} : (\mathsf{F}_0 \wr \Sigma)^{\times n} \xrightarrow{(\mathsf{F}_0 \wr Y_i)^{\times n}} (\mathsf{F}_0 \wr \mathcal{V}^{op})^{\times n} \xrightarrow{\otimes^n} (\mathcal{V}^{op})^{\times n} \xrightarrow{\otimes} \mathcal{V}^{op}. \tag{3.14}$$

Finally we note that, after including the omitted passive information (namely the $(Y_i)^{\otimes}$), Diagram (3.13) commutes (where empty 2-cells are the identity). We denote the composite of the completed top row in Diagram (3.13) by $N^1_{(Y_i),X}$, or just N^1 if the sequences are clear; note that the bottom row is precisely (the opposite of) $N^0_{(Y_i),\mathbb{F}X}$:

$$\lambda_1^n \mathbb{T}_0 \xrightarrow{\mathbb{V}} (\mathsf{F}_0 \wr \Sigma)^{\times n} \times \mathsf{F}_0 \wr \mathbb{T}_0 \longrightarrow (\mathsf{F}_0 \wr \Sigma)^{\times n} \times \mathsf{F}_0^{\wr 2} \Sigma \xrightarrow{(Y_i)^{\otimes} \times \mathsf{F}_0 \wr (\otimes \circ \mathsf{F}_0 \wr \mathbb{F}X)} \mathcal{V}^{op} \times \mathsf{F}_0 \wr \mathcal{V}^{op} \xrightarrow{\otimes} \mathcal{V}^{op}.$$

Two consecutive applications of Lemma A.1.8 imply the following results:

Lemma 3.3.15. *L_1 is the left Kan extension of the (opposite of the) outermost span in the following diagram.*

$$\begin{tikzcd}
\mathsf{F}_0 \wr \mathbb{T}_0 \arrow[r, "\mathsf{F}_0 \wr X^{\otimes}"] \arrow[d, "\mathsf{F}_0 \wr \Sigma"'] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r, "\otimes"] & \mathcal{V}^{op} \\
\mathsf{F}_0 \wr \Sigma \arrow[ur, "\mathsf{F}_0 \wr \mathbb{F}X"'] \arrow[urr, bend right, "L_1"'] & &
\end{tikzcd}$$

$\square$

Lemma 3.3.16. *L_2 is the left Kan extension of the (opposite of the) outermost span in the following diagram.*

$$\begin{tikzcd}
(\mathsf{F}_0 \wr \Sigma)^{\times n} \times \mathsf{F}_0 \wr (\mathsf{F}_0 \wr \mathbb{T}_0) \arrow[r, "(Y_i)^{\otimes} \times \mathsf{F}_0 \wr (\otimes \circ \mathsf{F}_0 \wr X^{\otimes})"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V}^{op} \times \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r, "\otimes"] & \mathcal{V}^{op} \\
(\mathsf{F}_0 \wr \Sigma)^{\times n} \times \mathsf{F}_0 \wr (\mathsf{F}_0 \wr \Sigma) \arrow[ur, "(Y_i)^{\otimes} \times \mathsf{F}_0 \wr L_1"'] \arrow[urr, bend right, "L_2"'] & &
\end{tikzcd}$$

$\square$

Lemma 3.3.17. *$N^0_{(Y_i),\mathbb{F}X}$ is equivalent to the left Kan extension of $N^1_{(Y_i),X}$ over d_1.*

Proof. This follows from Lemma 3.3.16 from Lemma A.1.9. Indeed, the functors $\mathsf{F_0} \wr (-)$ and $\mathcal{E} \times (-)$ preserve pullbacks, so the second square in Diagram (3.13), once the diagram is extended to include the passive nodes, is a pullback; the first square is a pullback by definition. $\square$

Hence, an application of Lemma 3.3.13 and Lemma A.0.2 yields the following:

Corollary 3.3.18. *The coproduct in operads $\amalg \mathbb{F} Y_i \amalg FX)$ is isomorphic to left Kan extension of $N^1_{(Y_i),X}$ over val.* $\square$

3.3.3 The Coproduct of a Free Operad and a Generic Operad

For this section, we will still have sequences Y_i, but now a generic operad $\mathcal{P}$ (taking the place of the sequence X). We will build a categorical description of $\amalg \mathbb{F} Y_i \amalg \mathcal{P}$ as a coequalizer

$$\mathbb{F}(\amalg Y_i \amalg \mathbb{F}\mathcal{P}) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathbb{F}(\amalg Y_i \amalg \mathcal{P}) \xrightarrow{d} \amalg \mathbb{F} Y_i \amalg \mathcal{P} \tag{3.15}$$

As in Lemma 3.2.11, this will be accomplished by constructing an appropriate *split* coequalizer in $\mathsf{WSpan}(\Sigma, \mathcal{V})$ and identifying its image under the left Kan extension, using Proposition A.1.2.

Proposition 3.3.19. *The maps*

$$\begin{tikzcd}
\lambda_1^n \mathbb{T}_1^{op} \arrow[r, "N^1_{(Y_i),\mathcal{P}}"] \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_0"] & \mathcal{V} \\
\lambda_1^n \mathbb{T}_0^{op} \arrow[ur, "N^0_{(Y_i),\mathcal{P}}"'] \arrow[d, "(-)^{alt}"'] & \\
\lambda_1^n \mathbb{T}_{-1}^{op} \arrow[uur, bend right, "N^{-1}_{(Y_i),\mathcal{P}}"'] &
\end{tikzcd}$$

are part of a (split) coequalizer in $\mathsf{WSpan}(\Sigma, \mathcal{V})$.

Proof. This data will be build directly out of the split coequalizer data from Lemma 3.2.11. For completeness and clarity, we will still build each piece individually:

(1) $d_0 = (d_0, d_0'')$:

Consider the following diagram:

$$\begin{tikzcd}
\lambda_1^n \mathbb{T}_1 \arrow[r, "\mathbb{V}_a"] \arrow[d, "d_0"'] & \mathsf{F}_0 \wr \mathbb{T}_1 \arrow[r, "\mathsf{F}_0 \wr\ (\mathsf{F}_0 \wr \otimes\ \circ\ \mathsf{F}_0{}^{\wr 2} N_{\mathcal{P}}\ \circ\ \mathsf{F}_0 \wr \mathbb{V})"] \arrow[d, "\mathsf{F}_0 \wr d_0"'] \arrow[dr, phantom, "\Downarrow \mathsf{F}_0 \wr d_0'"] & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r, "\otimes"] & \mathcal{V}^{op} \\
\lambda_1^n \mathbb{T}_0 \arrow[r, "\mathbb{V}_a"] & \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[ur, bend right, "\mathsf{F}_0 \wr N_{\mathcal{P}}"'] & &
\end{tikzcd}$$

This extends uninterestingly to the passive vertices, via $(Y_i)^{\otimes}$. Call the *opposite* of this extended diagram d_0''.

(2) $d_1 = (d_1, d_1'')$:

Similarly, consider the diagram

$$\begin{tikzcd}
\lambda_1^n \mathbb{T}_1 \arrow[r, "\mathbb{V}_a"] \arrow[d, "d_1"'] & \mathsf{F}_0 \wr \mathbb{T}_1 \arrow[r, "\mathsf{F}_0 \wr\ (\mathsf{F}_0 \wr \otimes\ \circ\ \mathsf{F}_0{}^{\wr 2} N_{\mathcal{P}}\ \circ\ \mathsf{F}_0 \wr \mathbb{V})"] \arrow[d, "\mathsf{F}_0 \wr d_1"'] \arrow[dr, phantom, "\Downarrow \mathsf{F}_0 \wr d_1'"] & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r, "\otimes"] & \mathcal{V}^{op} \\
\lambda_1^n \mathbb{T}_0 \arrow[r, "\mathbb{V}_a"] & \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[ur, bend right, "\mathsf{F}_0 \wr N_{\mathcal{P}}"'] & &
\end{tikzcd}$$

Again, this extends uninterestingly to the passive vertices, and we call the opposite of this extended diagram d_1''.

(3) $d = ((-)^{alt}, d'')$:

Consider the following diagram:

$$\begin{tikzcd}
\lambda_1^n \mathbb{T}_0 \arrow[r, "\mathbb{V}_a"] \arrow[d, "(-)^{alt}"'] & \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[r, "\mathsf{F}_0 \wr N_{\mathcal{P}}"] \arrow[d, "\mathsf{F}_0 \wr val"'] \arrow[dr, phantom, "\Downarrow \mathsf{F}_0 \wr \tilde{\mu}"] & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r, "\otimes"] & \mathcal{V}^{op} \\
\lambda_1^n \mathbb{T}_{-1} \arrow[r, "\mathbb{V}_a"] & \mathsf{F}_0 \wr \Sigma \arrow[ur, bend right, "\mathsf{F}_0 \wr \mathcal{P}"'] & &
\end{tikzcd}$$

In the above, note that the left squares all commute. Moreover, since $(-)^{alt}$ equalizes d_1 and d_0 while $\tilde{\mu}$ equalizes d_0' and d_1' (by Lemma 3.2.11), we have that d coequalizers d_0 and d_1.

(4) $s = (s, id)$:

Consider the diagram,

$$\begin{tikzcd} \lambda_1^n\mathbb{T}_{-1} \arrow[r, "\mathbb{V}_a"] \arrow[d, "s"'] & \mathsf{F_0} \wr \Sigma \arrow[d, "\mathsf{F_0}\wr C_{(-)}"'] \arrow[rr, bend left, "\mathsf{F_0}\wr\mathcal{P}"] & & \mathsf{F_0} \wr \mathcal{V}^{op} \arrow[r, "\otimes"'] & \mathcal{V}^{op} \\ \lambda_1^n\mathbb{T}_0 \arrow[r, "\mathbb{V}_a"'] & \mathsf{F_0} \wr \mathbb{T}_0 \arrow[rru, "\mathsf{F_0}\wr N_{\mathcal{P}}"'] & & & \end{tikzcd}$$

and note that is commutes. Again, this extends uninterestingly to the passive vertices, where it still commutes, as does its opposite diagram; hence $s = (s, id)$ is well-defined. As we already have that the functor s is a section of the $(-)^{alt}$, and since $\tilde{\mu}$ is the identity of corollas (as in Lemma 3.2.11), (s, id) is a section of d.

(5) $s_0 = (s_0, id)$:

Define $s_0 : \lambda_1^n\mathbb{T}_0 \to \lambda_1^n\mathbb{T}_1$ by mapping $(T, (S_v))$ to the trivial iterated assembly data $(T, (S_v \to S_v))$. Clearly the functor s_0 is a section of d_1 and d_0; in fact, $N^0 = N^1 \circ s_0''$. As before, since $\tilde{\mu}$ remains the identity on corollas, the arrow (s_0, id) is a section of (d_0, d_0'') and (d_1, d_1'').

(6) $s_{-1} = (s_{-1}, id)$:

Define $s_{-1}(T, (S_v)) = (T, (T_v \to S_v))$ to be the co-trivial iterated assembly data. Again, that this is a section of (d_0, d_0'') follows from the non-labeled case.

Lastly, we need to verify that $d_1 s_{-1} = sd$. As in Lemma 3.2.11, we have to reconcile an additional $\mathsf{F_0} \wr (-)$, and again, this is only recording an indexing over a singleton set, and hence does not affect the natural transformations.

Thus, we have a strict coequalizer diagram, and hence a coequalizer in $\mathsf{WSpan}(\Sigma, \mathcal{V})$, as desired. $\square$

By Corollary A.1.2, each map above induces a map between the associated left Kan extensions. By Corollary 3.3.18 and Lemma 3.3.13, we have induced maps

$$d_0, d_1 : \amalg \mathbb{F} Y_i \amalg \mathbb{F}\mathbb{F}\mathcal{P} \to \amalg \mathbb{F} Y_i \amalg \mathbb{F}\mathcal{P})$$

and, since split coequalizers are absolute, by Proposition 3.3.19 the induced map

$$d : \amalg \mathbb{F} Y_i \amalg \mathbb{F}\mathcal{P} \to \mathrm{Lan}_{val}(N^{-1}_{(Y_i),\mathcal{P}})$$

is a coequalizer:

Corollary 3.3.20. $\mathrm{Lan}_{val}(N^{-1}_{(Y_i),\mathcal{P}})$ *is the coequalizer of the pair of maps*

$$d_0, d_1 : \amalg \mathbb{F} Y_i \amalg \mathbb{F}\mathbb{F}\mathcal{P} \rightrightarrows \amalg \mathbb{F} Y_i \amalg \mathbb{F}\mathcal{P}.$$

□

Finally, we have our proof of Proposition 3.3.6:

proof of Proposition 3.3.6. This follows immediately from Corollary 3.3.20 and the canonical coequalizer description of an algebra over a monad as seen in Diagram (3.15). □ □

Remark 3.3.21 (The Coproduct of Two Generic Operads)**.** The coproduct of (singled coloured) operads has been discussed in many places, including [BO15]. The author expects to be able to construct this object with technology similar to the above. There are, however, a couple of key technical issues that are more complicated in this scenario, namely controlling the unit and the iterated mixed identifications of trees, and these have not yet been properly tackled. This, along with the W-construction, will be the subject of a sequel.

3.4 Cellular Extensions

Consider the following special case of a pushout in the category of operads:

Definition 3.4.1. Let $\mathcal{P}$ be an operad, with $u : Y_0 \to Y_1$ and $h : Y_0 \to \mathcal{P}$ maps of symmetric sequences. The following pushout in operads

$$\begin{tikzcd}
\mathbb{F}Y_0 \arrow[r] \arrow[d, "\mathbb{F}u"'] & \mathcal{P} \arrow[d] \\
\mathbb{F}Y_1 \arrow[r] & \mathcal{P}[u]
\end{tikzcd}$$

is call the *cellular extension* of $\mathcal{P}$ over u and h, as is denoted $\mathcal{P}[u]$.

Continuing to generalize the technology from the previous sections, we will build this operation as a left Kan extension. Pushouts of this form have been studied in many settings, since D. M. Kan [Hir03] showed that understanding these pushouts was the key to lifting modeling structures to algebras over a monad. This has been explored and exploited many times; for example, by Schwede-Shipley [SS00, Lemma 6.2] filtering cellular extensions over the commutative monoid monad, Spitzweck [Spi01] and Berger-Moerdijk [BM03] studying operads, White [Whi14a, Whi14b] and White-Yau [WY15] studying (algebras over) coloured operads, Harper [Har09, Har10], Pereira [Per16] and Pereira-Hausmann [HP15] studying spectral operads, along with many others.

Following in the footsteps of the many references above, we will then construct a filtration of the map $\mathcal{P} \to \mathcal{P}[u]$ in the underlying category of sequences.

Many aspects of the discussion below can be found in [BM03]. Specifically, Berger-Moerdijk filtered precisely these pushouts above in the Appendix of [BM03]. We restructure the results on a more categorically rigorous footing, and build a structure that may be generalized to different settings (see, e.g. Chapter 6). Additionally, the organization of this filtration is a generalization of the proof of Proposition 5.20 in [Per16].

To begin, we recall that the pushout defining the cellular extension is equivalent to the coequalizer

$$\mathcal{P}[u] \simeq \operatorname{coeq}\left(\mathbb{F}Y_0 \amalg \mathbb{F}Y_1 \amalg \mathcal{P} \overset{u_*}{\underset{h_*}{\rightrightarrows}} \mathbb{F}Y_1 \amalg \mathcal{P}\right)$$

where u_* and h_* are induced by u and h respectively.

Previously, we were able to realize this coequalizer in categories, before applying Lan.

However, in this case, we will not have that flexibility. Instead, we will use the universal property of the desired coequalizer, as described in Lemma A.1.7.

We begin by constructing the above maps categorically via Corollary A.1.2, as before.

We first note that the left Kan extension diagrams below are two particular cases of Corollary 3.3.6:

$$\begin{tikzcd} \lambda_1^2\mathbb{T}_{-1}^{op} \arrow[r, "N^{-1}_{(Y_0,Y_1),\mathcal{P}}"] \arrow[d, "val"'] & \mathcal{V} \\ \Sigma^{op} \arrow[ur, "\mathbb{F}Y_0 \amalg \mathbb{F}Y_1 \amalg \mathcal{P}"'] & \end{tikzcd} \qquad \begin{tikzcd} \lambda_1^1\mathbb{T}_{-1}^{op} \arrow[r, "N^{-1}_{(Y_1),\mathcal{P}}"] \arrow[d, "val"'] & \mathcal{V} \\ \Sigma^{op} \arrow[ur, "\mathbb{F}Y_1 \amalg \mathcal{P}"'] & \end{tikzcd}$$

We will think of $\lambda_1^2\mathbb{T}_{-1}$ as controlling all three labelings, while $\lambda_1^1\mathbb{T}_{-1}$ will only have p_1 passive nodes. Now, consider the following maps in $\mathsf{WSpan}(\Sigma, \mathcal{V})$.

(1) $d_u = (d_u, \Phi_u)$:

There is an obvious map $d_u : \lambda_1^2\mathbb{T}_{-1} \to \lambda_1^1\mathbb{T}_{-1}$ which is the identity on the underlying trees, but changes the labels of all p_0-nodes to p_1. Moreover, we have the following natural transformation:

$$\begin{tikzcd} \lambda_1^2\mathbb{T}_{-1} \arrow[r, "\mathbb{V}"] \arrow[d, "d_u"'] & (\mathsf{F}_0 \wr \Sigma)^{\times 2} \times \mathsf{F}_0 \wr \Sigma \arrow[r, "\mathsf{F}_0 \wr Y_0 \times \mathsf{F}_0 \wr Y_1 \times \mathsf{F}_0 \wr \mathcal{P}"] \arrow[d, "\amalg \times id"'] & \mathsf{F}_0 \wr \mathcal{V}^{op} \times \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \arrow[d, equal] \\ \lambda_1^1\mathbb{T}_{-1} \arrow[r, "\mathbb{V}"] & \mathsf{F}_0 \wr \Sigma \times \mathsf{F}_0 \wr \Sigma \arrow[r, "\mathsf{F}_0 \wr Y_1 \times \mathsf{F}_0 \wr \mathcal{P}"] \arrow[ur, phantom, "\Downarrow u_* \times \mathsf{F}_0 \wr \mathcal{P}"] & \mathsf{F}_0 \wr \mathcal{V}^{op} \times \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \end{tikzcd}$$

where u_* on the pair $((A, f_A), (B, f_B))$ is given by

$$\bigotimes_A Y_0(f_A(a)) \otimes \bigotimes_B Y_1(f_B(b)) \xrightarrow{\otimes u(f_A(a)) \otimes id} \bigotimes_A Y_1(f_A(a)) \otimes \bigotimes_B Y_1(f_B(b)).$$

Abusing notation, u_* will refer to this map and the map induced after applying Lan.

(2) $d_h = (d_h, \Phi_h)$:

The functor d_h will necessarily be more complicated, as the obvious functor "turn all p_0-nodes active" does not land in alternating trees. However, we can circumvent this issue by passing to $\lambda^n\mathbb{T}_0$. Define $\Delta_h : \lambda^3\mathbb{T}_0 \to \lambda^2\mathbb{T}_0$ by sending (T, λ) to $(T, \delta_h\lambda)$ where $\delta_h : \{p_0, p_1, a\} \to \{p_1, a\}$ is the identity on p_1 and a, and sends p_0 to a. Now, define d_h as the composite

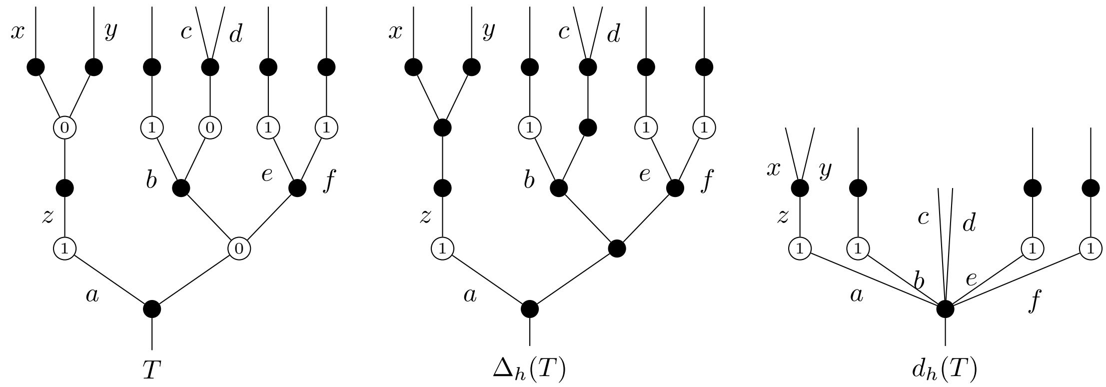
$$\lambda_1^2\mathbb{T}_{-1} \xhookrightarrow{s} \lambda_1^2\mathbb{T}_0 \simeq \lambda^3\mathbb{T}_0 \xrightarrow{\Delta_h} \lambda^2\mathbb{T}_0 \xrightarrow{\simeq} \lambda_1^1\mathbb{T}_0 \xrightarrow{d_1} \lambda_1^1\mathbb{T}_{-1}.$$

Heuristically, this mapping first changes the labelings, and then collapses all the new connected components of active nodes together.

We define Φ_h as the following commutative diagram, where empty 2-cells are the identity;

$$\begin{tikzcd}
\lambda_1^2\mathbb{T}_{-1} \arrow[r] \arrow[d, "s"'] & (\mathsf{F}_0 \wr \Sigma)^{\times 2} \times \mathsf{F}_0 \wr \Sigma \arrow[r] \arrow[d, "\mathsf{F}_0\wr C_{(-)}"'] \arrow[dr, "\sigma^1"] & (\mathsf{F}_0 \wr \mathcal{V}^{op})^{\times 2} \times \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[rr] \arrow[dr, "\sigma^1"] & & (\mathcal{V}^{op})^{\times 3} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\lambda_1^1\mathbb{T}_0 \arrow[r] \arrow[d, "\simeq"'] & (\mathsf{F}_0 \wr \Sigma)^{\times 2} \times \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[r] \arrow[d, "\mathrm{coll}"] & (\mathsf{F}_0 \wr \Sigma)^{\times 2} \times \mathsf{F}_0^{\wr 2}\Sigma \arrow[r] & (\mathsf{F}_0 \wr \mathcal{V}^{op})^{\times 2} \times \mathsf{F}_0^{\wr 2}\mathcal{V}^{op} \arrow[r] \arrow[d, "\mathrm{coll}"'] \arrow[dr, phantom, "\Updownarrow\Phi_\otimes"] & (\mathcal{V}^{op})^{\times 3} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\lambda^3\mathbb{T}_0 \arrow[r] \arrow[d, "\Delta_h"'] & (\mathsf{F}_0 \wr \Sigma)^{\times 2} \times \mathsf{F}_0 \wr \Sigma \arrow[rr] \arrow[d, "\amalg"] & & (\mathsf{F}_0 \wr \mathcal{V}^{op})^{\times 2} \times \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\amalg"'] & (\mathcal{V}^{op})^{\times 3} \arrow[d, "\otimes\times id"'] & \mathcal{V}^{op} \arrow[d, equal] \\
\lambda^2\mathbb{T}_0 \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\Updownarrow\Phi_p"] & \mathsf{F}_0 \wr \Sigma \times \mathsf{F}_0 \wr \Sigma \arrow[rr] & & \mathsf{F}_0 \wr \mathcal{V}^{op} \times \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r] \arrow[dr, phantom, "\Updownarrow\Phi_\otimes^{-1}"] & (\mathcal{V}^{op})^{\times 2} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\lambda_1^1\mathbb{T}_0 \arrow[r] \arrow[d, "d_1"'] & \mathsf{F}_0 \wr \Sigma \times \mathsf{F}_0 \wr \Omega_0 \arrow[r, "\mathbb{V}"] \arrow[d] \arrow[drr, "\mathsf{F}_0\wr Y_1 \times \mathsf{F}_0\wr\tilde{\mu}"] & \mathsf{F}_0 \wr \Sigma \times \mathsf{F}_0^{\wr 2}\Sigma \arrow[r] \arrow[ul, "\mathrm{coll}"'] & \mathsf{F}_0 \wr \mathcal{V}^{op} \times \mathsf{F}_0^{\wr 2}\mathcal{V}^{op} \arrow[r] \arrow[u, "\mathrm{coll}"] \arrow[d, "\sigma^0"] & (\mathcal{V}^{op})^{\times 2} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\lambda_1^1\mathbb{T}_{-1} \arrow[r] & \mathsf{F}_0 \wr \Sigma \times \mathsf{F}_0 \wr \Sigma \arrow[rr] & & \mathsf{F}_0 \wr \mathcal{V}^{op} \times \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r] & (\mathcal{V}^{op})^{\times 2} \arrow[r] & \mathcal{V}^{op}
\end{tikzcd}$$

Example 3.4.2. We demonstrate this functor on an example element in $\lambda_1^2\mathbb{T}_{-1}$ below. The source object T is on the left; the (non-alternating) tree $\Delta_h(T)$ which has converted the p_0-nodes active is in the middle; and the right tree has collapsed all the connected active components. As before, black nodes are active, while white are passive, and we have labeled the interior of the node to denote which kind of passive it is.



It is now clear why we cannot build a categorical equalizer of these two maps: objects would have to be equivalences classes of two-coloured trees, among other things. However, we will instead build a category (more precisely, an object in $\mathsf{WSpan}(\Sigma, \mathcal{V})$) which realizes this coequalizer after taking Lan, by building in extra maps which exactly make the identifications we desire.

Definition 3.4.3. Let $\mathbb{T}_e$ ("e" for "extension") be the category with objects $(T, \lambda) \in \mathrm{Ob}(\lambda_1^2\mathbb{T}_{-1})$, and with arrows generated by:

(1) isomorphisms in $\lambda_1^2\mathbb{T}_{-1}$;

(2) "passive relabeling" maps Δ_v: identity map on the underlying tree, but change the labeling on a vertex $v \in V(T)$ from p_1 to p_0;

(3) "active substitution" maps θ_{w,S^w}: underlying inner face map, which take an active node T_w and substitute a (p_0, a)-alternating height 3 tree S^w such that $d_h(S^w) = T_w$.

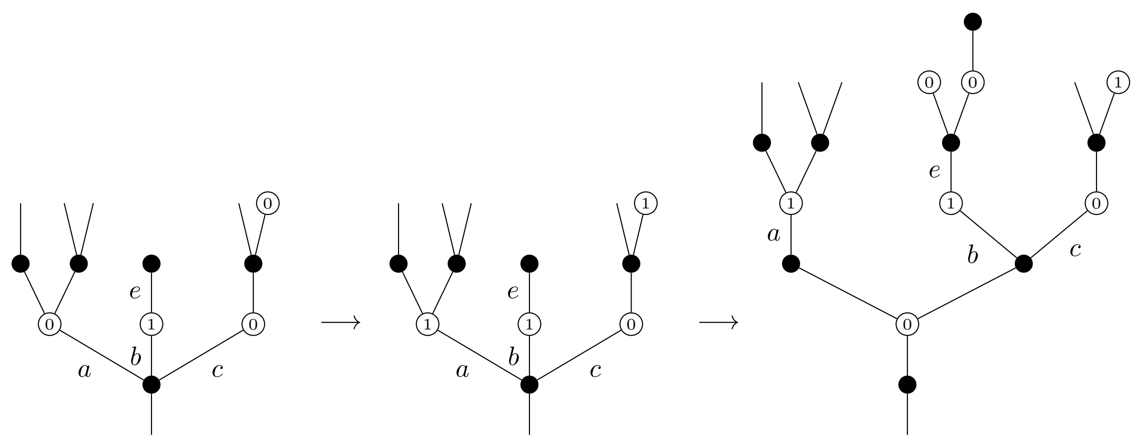
Relations are generated by the following:

(1) any two passive relabeling maps commute;

(2) compositions of active substitutions are identified if they induce the same underlying map in $\mathbb{T}_0$ after forgetting labels; and

(3) if δ_v and θ_{w,S^w} are passive relabeling and active substitution maps respectively, with $v, w \in V(T)$, then $\Delta_v \theta_{w,S_w} = \theta_{w,S_w} \Delta_v$.

Example 3.4.4. The following is a composite of passive relabelings followed by some active substitutions:



Remark 3.4.5. Some remarks:

(1) We observe that any map f has a unique decomposition (up to isomorphism) as $f = \Theta\Delta = \Delta\Theta$, where Δ (respectively Θ) is a composite of passive relabeling maps (respectively, active substitution maps).

(2) If we have a map $\Theta : S \to T$, then $d_h(S) = d_h(T)$.

(3) $\lambda_1^2 \mathbb{T}_{-1}$ embeds into $\mathbb{T}_e$ as a faithful subcategory, and *val* extends over this inclusion (as inner face maps are tall).

We extend our nerve evaluation functors to this category, on the condition that our preferred sequence X is again actually an operad $\mathcal{P}$:

Definition 3.4.6. Define $N^e_{(Y_0,Y_1),\mathcal{P}} = N^e : \mathbb{T}_e^{op} \to \mathcal{V}$ as $N^{-1}_{(Y_0,Y_1),\mathcal{P}}$ on objects and isomorphisms. On passive relabeling maps δ_v, N^e is given by an application of u in the v-th component; on active substitution maps θ_{w,S_w}, it is given by an application of h in the appropriate components, followed by the operad structure map $\tilde{\mu}_{S_w}$ on those same components.

Note that N^e extends N^{-1} over the faithful inclusion mentioned above.

While $\mathbb{T}_e$ will provide us with the most flexibility for building our desired filtration, the proof that this category indexes cellular extensions requires one more finality result.

Definition 3.4.7. Let $\mathbb{T}_e^0$ denote the full subcategory of $\mathbb{T}_e$ spanned by those trees T with $|T|_{p_0} = 0$. More, let $\vec{\mathbb{T}_e}$ denote the wide subcategory of $\mathbb{T}_e$ which has

(1) all isomorphisms;

(2) maps $d_h : d_h(T) \to T$; and

(3) maps $d_u : d_u(T) \to T$.

We note that, by construction, the source of any non-invertible map in $\vec{\mathbb{T}_e}$ is in $\mathbb{T}_e^0$.

Lemma 3.4.8. *For all n, $\vec{\mathbb{T}}_e^{op} \downarrow n$ is final in $\mathbb{T}_e^{op} \downarrow n$.*

Proof. Since $\vec{\mathbb{T}_e}$ is a wide subcategory, it is clear that each relevant overcategory $(\vec{\mathbb{T}}_e^{op} \downarrow n) \downarrow (T \downarrow n)$ is inhabited. Further, we observe that to show such an overcategory is connected, it suffices to show that any map $f : S \to T$ in $\mathbb{T}_e$ factors as a zig-zag of maps in $\vec{\mathbb{T}_e}$. Given such a map f, we may factor f in $\mathbb{T}_e$ as

$$f : S \xrightarrow{\Delta_h} S' \xrightarrow{\Theta_u}$$

where Δ_h is a composite of passive relabelings, and Θ_u a composite of active substitutions. Note then, by 3.4.5, that $d_h(S) = d_h(S')$ and $d_u(T) = d_u(S')$. Thus, we have our zig-zag factorization

$$T \xleftarrow{d_u} d_uT = d_uS' \xrightarrow{d_u} S' \xleftarrow{d_h} d_hS' = d_hS \xrightarrow{d_h} S,$$

completing the argument. $\square$

One final piece of notation for this subsection. If Ξ denotes any of our categories of trees constructed in this section, let $\Xi(n)$ denote the full subcategory of trees with n leaves.

This leads us to our categorical description of $\mathcal{P}[u]$:

Proposition 3.4.9. $\mathrm{Lan}_{val} N^e$ *is isomorphic to the coequalizer of the pair*

$$(d_u, d_h) : \mathbb{F}Y_0 \amalg \mathbb{F}Y_1 \amalg \mathcal{P} \simeq \mathrm{Lan}_{val} N^{-1}_{(Y_0,Y_1),\mathcal{P}} \rightrightarrows \mathrm{Lan}_{val} N^{-1}_{(Y_1),\mathcal{P}} \simeq \mathbb{F}Y_1 \amalg \mathcal{P};$$

In particular, $\mathrm{Lan}_{val} N^e \simeq \mathcal{P}[u]$.

Proof. By Lemma 3.4.8, we have

$$\mathrm{Lan}_{val} N^e(n) \simeq \operatorname*{colim}_{\substack{\mathbb{T}_e^{op}\downarrow n \\ val(T)\leftarrow n}} N^e(T) \simeq \operatorname*{colim}_{\substack{\vec{\mathbb{T}}_e^{op}\downarrow n \\ val(T)\leftarrow n}} N^e(T)$$

Since $\mathbb{T}_e^0$ contains all the minimal elements of $\vec{\mathbb{T}}_e$, we may write

$$\mathrm{Lan}_{val} N^e(n) \simeq \coprod_{\mathbb{T}_e^0(n)} N^e(S) \otimes_{\mathrm{Aut}(S)} \Sigma_n / \sim$$

for relations $\sim$ generated by the morphisms in $\vec{\mathbb{T}}_e$. Specifically, we that $\mathrm{Lan}_{val} N^e$ has the following universal property: for any $Z \in \mathcal{V}^{\Sigma^{op}}$, a map $\mathcal{V}^{\Sigma^{op}}(\mathrm{Lan}_{val} N^e, Z)$ is given by a collection of maps of the form

$$f(T) : N^e(T) \otimes_{\mathrm{Aut}(T)} \Sigma_n \to Z(n)$$

indexed by the objects in $\vec{\mathbb{T}}_e(n)$, such that for any maps $\varphi : S \to T$ and $\varphi' : S' \to T$ in $\vec{\mathbb{T}}_e$, we have that the diagram

$$\begin{array}{ccc} N^e(T) & \xrightarrow{\varphi_*} & N^e(S) \\ \Big\downarrow{\scriptstyle \varphi'_*} & & \Big\downarrow{\scriptstyle f(S)} \\ N^e(S') & \xrightarrow{f(S')} & Z(n) \end{array}$$

commutes, with both compositions equal to $f(T)$. Since for every target T there are exactly two maps (up to isomorphism), d_h and d_u, in $\vec{\mathbb{T}}_e$, we conclude that map $\operatorname{Lan}_{val} N^e \to Z$ are given by collections $\{f(S)\}$, this time indexed over the elements of $\mathbb{T}_e^0$, such that for all Δ and Θ as before, we have that the diagram

$$\begin{tikzcd} N^e(T) \arrow[r, "\Delta"] \arrow[d, "\Theta"'] & N^e(S) \arrow[d, "\varphi(T)"] \\ N^e(S') \arrow[r, "\varphi(S')"] & Z(n) \end{tikzcd}$$

commutes. Unpacking the notation, this is precisely the condition for the desired coequalizer as given by Lemma A.1.7. $\square$

3.5 Filtration of Cellular Extensions

In the previous section, we built the cellular extension $\mathcal{P}[u]$ as a left Kan extension out of $\mathbb{T}_e$. Thus, if we can construct a filtration of $\mathbb{T}_e$, we will get a filtration of $\mathcal{P}[u]$.

To that end, we make the following definitions, beginning with establishing some notation.

Definition 3.5.1. Given $T \in \mathbb{T}_e$, define $|T|_{p_0}$, $|T|_{p_1}$, and $|T|_a$ to be the number of p_0-, p_1-, and a-labeled vertices, respectively. Further, define $|T| = |T|_p = |T|_{p_0} + |T|_{p_1}$.

Now, we begin to filter our category $\mathbb{T}_e$:

Definition 3.5.2. (1) Let $\mathbb{T}_e[\leq k]$ (respectively $\mathbb{T}_e[k]$) be the full subcategory of $\mathbb{T}_e$ spanned by trees T with $|T| \leq k$ (respectively, $|T| = k$).

(2) Let $\mathbb{T}_e^{--}[\leq k]$ (respectively $\mathbb{T}_e^{-}[k]$) be the full subcategory of $\mathbb{T}_e[\leq k]$ (respectively $\mathbb{T}_e[k]$) spanned by trees T with $|T|_{p_1} \neq k$.

(3) Let $\mathbb{T}_e^0[\leq k]$ (respectively $\mathbb{T}_e^0[k]$) be the full subcategory of $\mathbb{T}_e[\leq k]$ (respectively $\mathbb{T}_e[k]$) spanned by trees T with $|T|_{p_1} = k$.

Remark 3.5.3. The categories $\mathbb{T}_e[k]$ and $\mathbb{T}_e^{-}[k]$ have only very limited morphisms, as there cannot be any “active substitutions”. Thus, any map $S \to T$ included just changes some p_1-labelings into p_0-labelings, while the underlying alternating tree in $\lambda_1^1\mathbb{T}_{-1}$ remains fixed.

The next three lemmas will allow us to connect the various levels of the filtration, providing the necessary means to bootstrap out way up our construction.

Lemma 3.5.4. *The category $\mathbb{T}_e[\leq k-1]^{op} \downarrow n$ is terminal in $\mathbb{T}_e^-[\leq k]^{op} \downarrow n$*

Proof. We need to show that, for fixed $n \to val(T)$ with $T \in \mathbb{T}_e^-[\leq k]$, the overcategory $(\mathbb{T}_e[\leq k-1]^{op} \downarrow n) \downarrow (n \to val(T))$ is non-empty and connected. Since $|T|_{p_0} \neq 0$ by assumption, $d_h(T) \in \mathbb{T}_e[\leq k-1]$, and so we have the natural object

$$\begin{tikzcd} val(T) & & val(d_h(T)) \arrow[ll] \\ & n \arrow[ul] \arrow[ur] & \end{tikzcd}$$

in the desired overcategory. Moreover, given any other object

$$n \to val(S) \xrightarrow{val(f)} val(T)$$

in this overcategory, Remark 3.4.5 says we have a factorization $f : S \xrightarrow{\Delta} S' \xrightarrow{\Theta} T$, and moreover that $d_h(T) \to T$ factors through S':

$$\begin{tikzcd} & val(T) & \\ val(d_h(T)) \arrow[ur] \arrow[r, dashed, "val(\Theta')"] & val(S') \arrow[u, "val(\Theta)"'] & val(S) \arrow[l, "val(\Delta)"'] \arrow[ul, "val(f)"'] \\ & n \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

Thus this overcategory is connected, and hence the result is shown. $\square$

Lemma 3.5.5. $\mathbb{T}_e^0[k]^{op} \downarrow n$ *is terminal in* $\mathbb{T}_e[k]^{op} \downarrow n$.

Proof. Analogously to the above, we have that the arrow $d_u(T) \to T$ provides a canonical element in the necessary overcategory, and since the only maps in $\mathbb{T}_e[k]$ are of type (2), any map $\Delta : S \to T$ induces a factorization of $d_u(T) \to T$ through S. $\square$

Lemma 3.5.6. $\mathbb{T}_e[\leq k]$ *is the isomorphic to the pushout below.*

$$\begin{tikzcd}
\mathbb{T}_e^-[k] \arrow[r] \arrow[d] & \mathbb{T}_e^-[\leq k] \arrow[d] \\
\mathbb{T}_e[k] \arrow[r] & \mathbb{T}_e[\leq k]
\end{tikzcd}$$

In fact, it is a nervous pushout of fully-faithful functors (see A.1.4). Moreover, this result also holds if we restrict to the subcategories of trees with exactly n leaves.

Proof. Since maps in $\mathbb{T}_e$ can only increase $|-|$ by adding to $|-|_{p_0}$, if T is a tree with $|T|_{p+1} = k$, and S is the source (respectively, target) of an arrow to (respectively, from) T in $\mathbb{T}_e[\leq k]$, then $|S| = k$, and hence the arrow is actually in $\mathbb{T}_e[k]$. The result is then immediate from unpacking definitions. The moreover follows from the fact that no map in $\mathbb{T}_e$ changes the number of leaves. $\square$

Abusing notation, we will denote by N^e the restriction of that functor to any of the subcategories of $\mathbb{T}_e$ described above.

We now define the sequences which will make up our filtration of $\mathcal{P}[u]$:

Definition 3.5.7. Let $\mathcal{P}_k$ denote the left Kan extension

$$\begin{tikzcd}
\mathbb{T}_e[\leq k]^{op} \arrow[r, "N^e"] \arrow[d, "val"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\
\Sigma^{op} \arrow[ur, "\mathcal{P}_k"'] &
\end{tikzcd}$$

Note that by Lemma A.1.2, we have natural maps $\mathcal{P}_{k-1} \to \mathcal{P}_k$.

3.5.1 Notation

In order to state our filtration result, we will need to identify another categorical construction. This filtration will be built out of "pushout products over trees of maps of sequences". This subsection will be dedicated to making the components of that statement precise.

Recall the categorical wreath product, defined in Definition 3.1.2.

Definition 3.5.8. Given a map $u : Y_0 \to Y_1$ of sequences and $(A, D) \in \mathsf{F_0} \wr \Sigma$, we borrow notation from [BM03] and define the functor $[u]^D : (0 \to 1)^A \to \mathcal{V}$ as the composite

$$(0 \to 1)^A \to \mathsf{F_0} \wr V \xrightarrow{\otimes} \mathcal{V}$$

where the first map is defined on $\epsilon : A \to \{0, 1\}$ by

$$(\epsilon(a))_a \mapsto (A, (Y_{\epsilon(a)}(D(a)))_a).$$

We recall that, in a general category $\mathcal{C}$, a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is called *convex* if whenever $c' \in \mathcal{C}'$ and $c \mapsto c'$ is an arrow in $\mathcal{C}$, then both c' and the map are in $\mathcal{C}'$.

Definition 3.5.9. Let $\mathcal{C}$ be a convex subcategory of $(0 \to 1)^A$. We define $Q^A_{\mathcal{C}}[u]^D := \operatorname{colim}_{\mathcal{C}}[u]^D$; moreover, given nested convex subcategories $\mathcal{C}' \subseteq \mathcal{C}$, let

$$[u]^D \square^{\mathcal{C}}_{\mathcal{C}'} : Q^A_{\mathcal{C}'}[u]^D \to Q^A_{\mathcal{C}}[u]^D$$

denote the unique natural map.

In particular, if $\mathcal{C}$ is the full "punctured cube" subcategory $(0 \to 1)^A \backslash \{(1)_a\}$, we simplify the notation as follows:

$$Q[u]^D := Q_{\mathcal{C}}[u]^D$$

$$[u]^{\square D} := [u]^D \square^{(0\to 1)^A}_{\mathcal{C}} : Q[u]^D \to \bigotimes_{a \in A} Y_1(D(a)).$$

3.5.2 Filtration Result

We can now state our filtration of the cellular extension $\mathcal{P} \to \mathcal{P}[u]$:

Theorem 3.5.10. *Let $\mathcal{P}$ be an operad, and suppose we are given a map of symmetric sequences $u : Y_0 \to Y_1$. Then we have a levelwise filtration in sequences of the cellular* *extension $\mathcal{P} \to \mathcal{P}[u]$. Specifically, for each n, in the underlying category $\mathcal{V}^{\Sigma_n}$ we have a filtration*

$$\mathcal{P}[u](n) \cong \operatorname{colim}(\mathcal{P}_0(n) \to \mathcal{P}_1(n) \to \mathcal{P}_2(n) \to \dots)$$

of $\mathcal{P}(n) \to P[u](n)$, where $\mathcal{P}_0 := \mathcal{P}$ and the $\mathcal{P}_k$ are build inductively via pushout diagrams of the form

$$\begin{array}{ccc}
\displaystyle\coprod_{[T]\in\mathbb{T}_e^0[k](n)/\simeq} \left(\bigotimes_{v\in V_b(T)} \mathcal{P}(T_v) \otimes Q[u]^{\mathbb{V}_p(T)} \right) \otimes_{\mathrm{Aut}(T)} \Sigma_n & \longrightarrow & \mathcal{P}_{k-1}(n) \\
\downarrow & & \downarrow \\
\displaystyle\coprod_{[T]\in\mathbb{T}_e^0[k](n)/\simeq} \left(\bigotimes_{v\in V_b(T)} \mathcal{P}(T_v) \otimes \bigotimes_{v\in V_p(T)} Y_1(T_v) \right) \otimes_{\mathrm{Aut}(T)} \Sigma_n & \longrightarrow & \mathcal{P}_k(n)
\end{array}$$

where the left vertical map is the iterated box product

$$\coprod \mathop{\square}_{V_b(T)} \iota_{P(T_v)} \square [u]^{\square \mathbb{V}_p(T)}$$

where $\iota_{P(T_v)}$ denotes the canonical map $\varnothing \to \mathcal{P}(T_v)$ out of the initial object, and $\mathbb{T}_e^0[k](n)$ is as above (Definition 3.5.2).

Proof. Combining Lemmas A.1.5 and 3.5.6, we have that $\mathcal{P}_k(n)$ can be computed as the pushout

$$\begin{array}{ccc}
\operatorname{colim}_{\mathbb{T}_e^-[k]^{op}\downarrow n} N^e & \longrightarrow & \operatorname{colim}_{\mathbb{T}_e^-[\leq k]^{op}\downarrow n} N^e \\
\downarrow & & \downarrow \\
\operatorname{colim}_{\mathbb{T}_e[k]^{op}\downarrow n} N^e & \longrightarrow & \operatorname{colim}_{\mathbb{T}_e[\leq k]^{op}\downarrow n} N^e := \mathcal{P}_k(n)
\end{array} \tag{3.16}$$

By Lemma 3.5.4, the top right corner can be identified with $\mathcal{P}_{k-1}(n)$. Thus, it remains to identify the left hand side.

By Lemma 3.5.5, we may replace the bottom left corner with $\operatorname{colim}_{\mathbb{T}_e^0[k]^{op}\downarrow n} N^e$. Now, given $T \in \mathbb{T}_e^0[k]$, let $[T]$ denote the isomorphism class of T in $\mathbb{T}_e^0[k]$. With this notation, the bottom left corner can further be identified with

$$\coprod_{[T]\in\mathbb{T}_e^0[k](n)/\simeq} N^e(T)\otimes_{\mathrm{Aut}(T)}\Sigma_n = \coprod_{[T]}\left(\bigotimes_{v\in V_a(T)}\mathcal{P}(T_v)\otimes\bigotimes_{v\in V_{p_1}(T)}Y_1(T_v)\right)\otimes_{\mathrm{Aut}(T)}\Sigma_n.$$

Next, we observe that the non-invertible morphisms of $\mathbb{T}_e^-[k]^{op}\downarrow n$ are just those which change the labeling of some nodes from p_0 to p_1. Given S and T in $\mathbb{T}_e^-[k]$, write $S\sim T$ if they are in the same path component, and again note that this implies $|S|_p=|T|_p$, and moreover that S and T forget to the same object in $\lambda_1^1\mathbb{T}_{-1}$. Denote the path component of T by (T).

We note that the set of path components is equal to the set of isomorphism classes in $\mathbb{T}_e^0[k]$, as both are just determined by their forgetful image in $\lambda_1^1\mathbb{T}_{-1}$.

To account for the Σ_n-action on the indexing category, we note that each connected component of $\mathbb{T}_e^-[k]^{op}\downarrow n$ has an action of $\mathrm{Aut}([T])$. Thus, the top left corner of Diagram (3.16) can be identified with the image on the right below:

$$\begin{array}{c}\displaystyle\coprod_{[T]\in\mathbb{T}_e^0[k](n)/\sim}\left(\coprod_{S\in(T)\setminus\{T\}}N^e(S)\right)\otimes_{\mathrm{Aut}(T)}\Sigma n\\ \Big\downarrow{\scriptstyle\mathrm{colim}}\\ \displaystyle\coprod_{[T]}\left(\bigotimes_{v\in V_a(T)}\mathcal{P}(T_v)\otimes Q[u]^{\mathbb{V}_p(T)}\right)\otimes_{\mathrm{Aut}(T)}\Sigma_n\end{array}$$

where $Q[u]^{\mathbb{V}_p(T)}$ is the source of the pushout product map defined in Definition 3.5.9.

Lastly, this left-side map is induced, via Lemma A.1.2, by an inclusion of categories, in particular the product of multiple inclusions of categories, each corresponding the inclusion of a punctured cube into the full cube. Thus, after taking colimits, we have that the left-side map in Diagram (3.16) is in fact (multiple copies of) the pushout-product maps

$$[u]^{\square\mathbb{V}_p(T)}:Q[u]^{\mathbb{V}_p(T)}\to\bigotimes_{v\in V_p(T)}Y_1(T_v)),$$

as desired. $\square$

3.6 An Aside on the Composition Product

We end this chapter by briefly using the above notations and analysis to present the composition product description of operads in a new language. To do so, we build another category of structured trees.

Definition 3.6.1. Let $\mathbb{T}_0[\frac{2}{m}]$ denote the full subcategory of $\mathbb{T}_0$ height-2 trees with root vertex C_m; that is, all trees of the form $T = C_m \circ (C_{k_i})$.

This has a natural vertex functor

$$\mathbb{V} : \mathbb{T}_0[\tfrac{2}{m}] \to \Sigma_m \wr \Sigma$$

(where we are thinking of Σ_m as the full subcategory of F_0 spanned by $\{[m]\}$), sending $C_m \circ (C_{k_i})$ to the tuple $([m]; k_1, \ldots, k_n)$, and an underlying valence functor

$$val : \mathbb{T}_0[\tfrac{2}{m}] \to \Sigma.$$

Let $\mathbb{T}_0^-[\frac{2}{m}]$ denote the subcategory of $\mathbb{T}_0[\frac{2}{m}]$ where we restrict arrows to just vertex-preserving (non-planar) isomorphisms; in particular, the automorphism group of $T = C_m \circ (C_{k_i})$ is just $\Sigma_{k_1} \times \ldots \times \Sigma_{k_m}$ with no wreath products. In this case, $\mathbb{V}$ lands in simply $\Sigma^{\times m} \times \Sigma_m$.

Definition 3.6.2. Given symmetric sequences $Y_1, \ldots, Y_m$ and X, let $N^{m,-}_{(Y_i),X}$ denote the composite functor

$$N^m_{(Y_i),X} : \mathbb{T}_0[\tfrac{2}{m}]^{op} \xrightarrow{\mathbb{V}} (\Sigma^{\times m})^{op} \times \Sigma_m^{op} \xrightarrow{Y_1 \times \ldots \times Y_m \times X} \mathcal{V}^{\times m+1} \xrightarrow{\otimes} \mathcal{V}.$$

If the Y_i are all equal, then we have a similar functor $N^m_{(Y),X}$ given by

$$N^m_{(Y),X} : \mathbb{T}_0[\tfrac{2}{m}]^{op} \xrightarrow{\mathbb{V}} \Sigma_m \wr \Sigma^{op} \xrightarrow{\Sigma_m \wr Y \times X} \Sigma_m \wr \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}.$$

Lemma 3.6.3. *Suppose we are given symmetric sequences $Y_1, \ldots, Y_m$, and X in $\mathcal{V}^{\Sigma^{op}}$.*

(1) The tensor product $Y_1 \otimes \ldots \otimes Y_m$ is isomorphic to the left Kan extension $\mathrm{Lan}_{val}\, N^{m,-}_{(Y_i),*}$, where $*$ is the constant presheaf to the unit of $\mathcal{V}$.

$$\begin{tikzcd} \mathbb{T}_0^-[\tfrac{2}{m}]^{op} \arrow[r, "N^{m,-}_{(Y_i),*}"] \arrow[d, "val"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\ \Sigma^{op} \arrow[ur, "\mathrm{Lan} \simeq Y_1 \otimes \ldots \otimes Y_m"'] & \end{tikzcd}$$

(2) The composition product $X \circ Y$ is isomorphic to the left Kan extension $\mathrm{Lan}_{val}(\amalg N^m_{(Y),X})$;

$$\begin{tikzcd} \coprod_m \mathbb{T}_0[\tfrac{2}{m}]^{op} \arrow[r, "\amalg N^m_{(Y),X}"] \arrow[d, "val"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\ \Sigma^{op} \arrow[ur, "\mathrm{Lan} \simeq X \circ Y"'] & \end{tikzcd}$$

Proof. These follow formally from unpacking the explicit description and universal properties of the coends from Definition 2.2.6 and left Kan extensions. □

Remark 3.6.4. We expect to be able to show that the structure maps of being a monoidal product in $\mathcal{V}$ can be constructed in this language, similarly to the monad structure maps above, in fact using similar technology of categorical pullbacks. The precise formulation will be explored in a sequel.

Chapter 4

Equivariant Homotopy Theory and Equivariant Operads

One of the major goals of rebuilding the construction of general operads given in the first chapters of this thesis is to provide categorical frameworks which are robust enough to be generalized in many different directions, particularly to the world of *equivariant* operads.

In this chapter, we begin by discussing some of the basics of equivariant homotopy theory, and specifically the complexities of equivariant operads mentioned in the introduction. After remodeling the theory of G-coloured operads using the machinery from Chapter 3, we will end by giving partial results on the existence of genuine model structures on $\mathcal{V}\mathsf{Op}^G_{\{*\}}$ for general $\mathcal{V}$. In particular, we will prove a conjecture of Blumberg-Hill comparing "N_∞-operads" with indexing systems.

To that end, we fix a finite group G, and a closed symmetric monoidal category $\mathcal{V}$.

Remark 4.0.1. While some of the following discussion may hold for compact Lie groups, much of it would have to be reworked significantly; see, e.g. [Blu06] for some examples of issues that can arise homotopically when working with infinite groups.

4.1 Introduction to Equivariant Homotopy Theory

We will briefly discus some of the major components of equivariant homotopy theory; see [Ada84] for a standard introduction, or [LMSM86] and [May96] for a comprehensive look.

Definition 4.1.1. Let F^G denote the category of *G-sets* and *G-maps*, and O_G the full subcategory of F^G spanned by the transitive G-sets G/H for H a subgroup of G.

We think of G also as a groupoid with one object and the set G of morphisms. More generally, given any G-set A, let $B_A G$ denote the groupoid with objects A and morphisms $g : a \to g.a$ for all pairs $(g, a) \in G \times A$.

Definition 4.1.2. Given any category $\mathcal{V}$, we denote by $\mathcal{V}^G$ the category of left G-objects and G-maps; equivalently, functors $G^{op} \to \mathcal{V}$.

Lemma 4.1.3. *If H is a subgroup of G, the natural map $H \to B_{G/H}G$ sending $* \mapsto eH$ induces an equivalence of categories $\mathcal{V}^H \leftrightarrows \mathcal{V}^{B_{G/H}G}$.*

Definition 4.1.4. For a subgroup H of G, define the *H-fixed point functor* $(-)^H : \mathcal{V}^G \to \mathcal{V}$ by the composition

$$\mathcal{V}^G \to \mathcal{V}^H \xrightarrow{\lim} \mathcal{V}$$

of the restriction functor with the limit functor. Considered all together, this provides a functor

$$\mathcal{V}^G \to \mathcal{V}^{O_G^{op}}$$

where $\mathcal{V}^{O_G^{op}}$ is the category of *G-coefficient systems in* $\mathcal{V}$, sending V to its *fixed-point system.*

Example 4.1.5. For $\mathcal{V}$ the category of spaces, simplicial sets, vector spaces, etc, this gives the usual notion. In particular, if $A \in \mathsf{F}^G$, then $A^H = \{a \in A \mid h.a = a \text{ for all } a \in A\}$.

Analogous to the fact that any cocomplete category $\mathcal{V}$ is copowered over sets, any $\mathcal{V}^G$ is copowered over G-sets: given $A \in \mathsf{F}^G$ and $V \in \mathcal{V}$, define $A \cdot V = \amalg_A V$, where the G-action acts both on the indices and the object V. This in fact defines a left adjoint to the fixed-point functors above:

Lemma 4.1.6. *The functors $(-)^H : \mathcal{V}^G \leftrightarrows \mathcal{V} : G/H \cdot (-)$ form an adjoint pair.* $\square$

Moreover, if $\mathcal{V}$ is enriched over $\mathcal{E}$ (e.g. $\mathcal{E} = \mathsf{Set}$), the category $\mathcal{V}^G$ is enriched over $\mathcal{E}^G$, as G acts on $\hom(V, W) \in \mathcal{E}$ by conjugation. Then we observe that $\mathcal{V}^G(V, W) = \mathcal{V}(V, W)^G$.

Given a homomorphism $\varphi : H \to G$, we have a collection of adjoints encoding a change of groups

$$\begin{tikzcd} \mathcal{V}^H \arrow[r, bend left, "\mathrm{ind}_H^G"] \arrow[r, bend right, "\mathrm{coind}_H^G"'] & \mathcal{V}^G \arrow[l, "\mathrm{res}_H^G"'] \end{tikzcd}$$

given by

$$\mathrm{ind}(V) = G \cdot_H V$$
$$\mathrm{coind}(V) = \mathrm{hom}^H(G, V)$$
$$\mathrm{res}(W) = \varphi^* W$$

Now, one of the first results in equivariant homotopy theory is Elmendorf's Theorem, which says that the "correct" homotopy theory of G-spaces ($\mathcal{V} = \mathsf{Top}$ or sSet) requires understanding the homotopy type of each fixed-point space.

Definition 4.1.7. A G-map $f : X \to Y$ of G-spaces is called a (genuine) *weak equivalence* (respectively, (genuine) *fibration*) if $f^H : X^H \to Y^H$ is so in Top for all $H \leq G$. We call f a (genuine) *cofibration* if it has the left lifting property against fibrations which are also weak equivalences.

In the category of coefficient systems, it is enough to consider the homotopy of the underlying objects:

Definition 4.1.8. A map of G-coefficient systems of spaces $f : A \to B$ is a *weak equivalence* (respectively, *fibration*) if $f(G/H) : A(G/H) \to B(G/H)$ is so in Top for all $H \leq G$.

Theorem 4.1.9. *[[Elm83, Pia91]] There are Quillen model structures on G-spaces and G-coefficient systems with the above weak equivalences and fibrations. Moreover, the adjunction* $(ev_{G/e}, \Phi)$ *is a Quillen equivalence.*

$$\Phi(-) : \mathsf{Top}^G \xrightarrow{\simeq_Q} \mathsf{Top}^{O_G^{op}} : ev_{G/e}.$$

The above model structure on Top^G is often referred to as the "genuine" model structure.

This definition of the homotopy theory of G-spaces is necessary to have, for example, Whitehead-type theorems.

Definition 4.1.10. A *G-CW complex* is a G-space built out of sequential pushouts of the form

$$\begin{tikzcd} \coprod_i G/H_i \cdot S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_i G/H_i \cdot D^n \arrow[r] & X_n. \end{tikzcd}$$

Theorem 4.1.11 ([LÖ5, Theorem 1.6])**.** *A map $f : X \to Y$ between G-CW complexes is a weak G-equivalence if and only if it is a* strong *G-homotopy equivalence: there eixsts a map $f' : Y \to X$ such that ff' and $f'f$ are G-homotopic to id_Y and id_X, respectively.*

The existence of this and similar results allows us to conclude that this notion of weak G-equivalence captures the correct homotopy theory of G-spaces.

Analogous results to Theorem 4.1.9 remain true for other base model categories $\mathcal{V}$ besides spaces: simplicial presheaves [Gui06], the Thomason model structure on small categories [BMO$^+$15], simplicial groups [Ste16], chain complexes [Ste16], and simplicial categories [Ber17], among many others. Much of this follows from the work of Piacenza [Pia91], Guillou [Gui06] and Stephan [Ste16], which says that model structures of that form on $\mathcal{V}^G$ exist given some assumptions of good behavior by the fixed-point functors on $\mathcal{V}^G$.

In fact, Stephan's results are slightly more general. We have already used the notation of the orbit category O_G of G, which has as objects all orbits G/H, and G-maps between them. Certain full subcategories of O_G are of particular importance.

Definition 4.1.12. Given a group G, we will denote by $\mathcal{L}(G)$ the lattice of subgroups of G under inclusions. For any subset $\mathcal{F} \subseteq \mathcal{L}(G)$, denote by $O_{\mathcal{F}}$ the full subcategory of O_G spanned by objects G/H with $H \in \mathcal{F}$.

A *family* of subgroups of G is a subset $\mathcal{F} \subseteq \mathcal{L}(G)$ of subgroups of G such that $\mathcal{F}$ is closed under subgroups and conjugation; that is, if $H \in \mathcal{F}$, then so is K^g for any $K \leq H$

and $g \in G$. Equivalently, $\mathcal{F} \subseteq \mathcal{L}(G)$ is a family if and only if the associated orbit category $O_{\mathcal{F}}$ is a *sieve* of O_G: for any arrow $A \to B$ in O_G with $B \in O_{\mathcal{F}}$, A is also in $O_{\mathcal{F}}$.

Now, let $\mathcal{V}$ be some cofibrantly generated model category, with generating cofibrations and trivial cofibrations I and J. We recall (cf. 1.2.8) that, for any family $\mathcal{F}$, $\mathcal{V}^{O_{\mathcal{F}}}$ admits the projective model structure.

Definition 4.1.13. Given a family $\mathcal{F}$, we say $\mathcal{V}^G$ admits the $\mathcal{F}$*-model structure* if there is a model structure on $\mathcal{V}^G$ where $f : A \to B$ is a weak equivalence (respectively, fibration) if $f^H : A^H \to B^H$ is so for all $H \in \mathcal{F}$.

Definition 4.1.14 ([Gui06, Ste16][BMO$^+$15, Proposition 1.5])**.** Let G be a group, and $\mathcal{F}$ a collection of subgroups of G. We call $\mathcal{V}$ $\mathcal{F}$*-cellular* if, for all $H \in \mathcal{F}$, the H-fixed point functor $(-)^H$

(1) preserves directed colimits of diagrams in $\mathcal{V}^G$, where each underlying arrow in $\mathcal{V}$ is a cofibration;

(2) preserves pushouts of diagrams where one leg is given by

$$G/K \otimes f : G/K \otimes A \to G/K \otimes B$$

for a subgroup $K \leq G$ and a generating cofibration $f : A \to B$ in $\mathcal{V}$; and

(3) for any subgroup $K \leq G$ and any object A of $\mathcal{V}$, the induced map

$$(G/K)^H \otimes A \to (G/K \otimes A)^H$$

is an isomorphism in $\mathcal{V}$.

There is a natural adjunction

$$\tau_* : \mathcal{V}^G \leftrightarrows \mathcal{V}^{O_{\mathcal{F}}^{op}} : \tau^*$$

induced by the inclusion of categories $\tau : G \hookrightarrow O_{\mathcal{F}}$ which sends the unique object to G/e. Explicitly, $\tau_* X(G/H) = X^H$, while $\tau^* Y = Y(G/e)$.

Proposition 4.1.15 ([Ste16, Proposition 2.6, Theorem 2.10])**.** *Let G be a group and $\mathcal{F}$ a collection of subgroups of G such that the cofibrantly generated model category $\mathcal{V}$ is $\mathcal{F}$-cellular. Then $\mathcal{V}^G$ admits the $\mathcal{F}$-model structure, where the generating cofibrations are given by*

$$I_{\mathcal{F}} = \{G/H \cdot i \mid H \in \mathcal{F},\ i \in I\}$$

and similarly for the generating trivial cofibrations. Moreover, the adjunction (τ^*, τ_*) is a Quillen equivalence between $\mathcal{F}$-coefficient systems with the projective model structure and G-objects with the $\mathcal{F}$-model structure.

Lastly, families also allow us to specify certain universal homotopy types:

Definition 4.1.16. Given a family $\mathcal{F}$ of subgroups of G, define a *universal space* for the family, if it exists, to be a space $E\mathcal{F}$ such that

$$E\mathcal{F}^H \sim \begin{cases} * & H \in \mathcal{F} \\ \varnothing & \text{otherwise} \end{cases}$$

Lemma 4.1.17 ([LÖ5])**.** *For all families $\mathcal{F}$, there exist models for $E\mathcal{F}$ which are G-CW complexes.*

4.2 Equivariant Operads

In this section, we will investigate the properties of the category of G-operads.

Definition 4.2.1. The category of *G-operads in $\mathcal{V}$* is the category $\mathcal{V}\mathsf{Op}^G$ of G-objects in $\mathcal{V}\mathsf{Op}$.

We will begin our discussion by restricting our consideration to just G-operads with a single colour. This subcategory $\mathcal{V}\mathsf{Op}^G_{\{*\}}$ is equivalent to the category $\mathcal{V}^G\mathsf{Op}_{\{*\}}$ of single-coloured operads in $\mathcal{V}^G$; that is, objects are symmetric sequences $\mathcal{O} = \{\mathcal{O}(n)\}$ of $G \times \Sigma_n$-objects $\mathcal{O}(n)$, with G-equivariant composition maps, and a G-fixed unit.

Definition 4.2.2. We denote the category of *symmetric G-sequences* $(\mathcal{V}^G)^{\Sigma^{op}} \simeq \mathcal{V}^{G\times\Sigma}$ by Sym^G.

4.2.1 Homotopical Considerations

Now, as we saw above in Section 2.2.3, the category $\mathcal{V}\mathsf{Op}_{\{*\}}$ of (single-coloured) $\mathcal{V}$-operads often has a model structure induced by the forgetful functor

$$\mathrm{fgt} : \mathcal{V}\mathsf{Op}_{\{*\}} \to \mathcal{V}^{\mathbb{N}}$$

where the category of non-symmetric sequences is given the projective model structure: an arrow $f : \mathcal{O} \to \mathcal{P}$ of $\mathcal{V}$-operads is a weak equivalence (respectively fibration) if $f(n) : \mathcal{O}(n) \to \mathcal{P}(n)$ is one in $\mathcal{V}$.

This leads us to a natural question: does the technology of [Ste16] endow the category of G-operads with a model structure, and if yes, it is a useful one?

Let us consider what this model structure would give us. We would be declaring that $f : \mathcal{O} \to \mathcal{P}$ is a weak equivalence of G-operads if $f^H : \mathcal{O}(n)^H \to \mathcal{P}(n)^H$ were weak equivalences in $\mathcal{V}$ for all $H \leq G$. In order for this to be a useful model structure, this should not, for example, identify any G-operads with substantially different algebras.

We return to the discussion in the introduction. Restricting to $\mathcal{V} = \mathsf{Top}$, we recall the E_∞-operads of Section 2.2.1, which encoded "commutative monoids up to coherent homotopy", providing models for infinite loop spaces, connective spectra, and commutative ring spectra; these are characterized by the fact that each space $\mathcal{O}(n)$ is Σ_n-free and contractible. In particular, if $\mathcal{O}$ is cofibrant in the above model structure, and the natural map $\mathcal{O} \to \mathsf{Comm}$ is a weak equivalence, then $\mathcal{O}$ is E_∞.

Equivariantly, there is a stronger notion of an infinite loop space:

Definition 4.2.3. Given a finite dimensional orthogonal G-representation V, let S^V denote the one-point compactification $V \amalg \{\infty\}$ of V; this is naturally a G-space, and we will call S^V a *representation sphere.*

Definition 4.2.4. A G-space X is called an *equivariant infinite loop space* if for all representations V there exists another G-space X_V such that $X \sim hom(S^V, X_V)$.

We again have a natural correspondence between equivariant infinite loop spaces and connective G-spectra. Moreover, we note that commutative G-monoids have unique maps not just of the form $X^n \to X$, but instead $N^A X \to X$, where A is an H-set for $H \leq G$, and $N^A X$ is a "multiplicative norm" of X, where G acts not only on the copies of X, but also on the *indexing* set A. In [HHR16], these unique (up to coherent homotopy) maps associated to "genuine commutative G-ring spectra" (up to coherent homotopy) were fundamental in the solution to the Kervaire Invariant One problem. As this structure is found on "universal deformations" of commutative monoids, again these objects should all be described by the same G-operad, as they were in the non-equivariant case.

Thus, we would like to generalize the notion of E_∞ so that it covers all of this additional behavior. A "natural" guess would be the following:

"A G-operad $\mathcal{O}$ is an E_∞-operad if each space $\mathcal{O}(n)$ is Σ_n-free and G-contractible."

Example 4.2.5. If $\mathcal{O}$ is a non-equivariant E_∞-operad, and we endow it with the trivial G-action, the resulting object has the $G \times \Sigma_n$-homotopy type given by

$$\mathcal{O}(n)^\Lambda \sim \begin{cases} * & \Lambda \leq H \\ \varnothing & \text{otherwise} \end{cases}$$

and in particular, it satisfies the above condition.

However, these operads encode structure *strictly weaker* than the above desiderata. In particular, such $\mathcal{O}$ only encode "infinite loop spaces with G-actions", or non-genuine commutative G-ring spectra with no norm maps [CW91, BH15].

Instead, the correct notion was later identified by Costenoble-Waner [CW91]. The main issue is that "Σ_n-free and G-contractible" does *not* determine a unique $G \times \Sigma_n$-homotopy type. In particular, it just tells us that

(1) if $\Lambda \leq G$, $\mathcal{O}(n)^\Lambda \simeq *$;

(2) if $\Lambda \cap \Sigma_n \neq \{e\}$, $\mathcal{O}(n)^\Lambda = \varnothing$.

If $H \leq G$ were the only subgroups of $G \times \Sigma_n$ that did not intersect Σ_n non-trivially, we would be done. However, there are often many more such subgroups:

Lemma 4.2.6 ([BH15])**.** *$\Lambda \leq G \times \Sigma_n$ is such that $\Lambda \cap \Sigma_n = \{e\}$ if and only if $\Lambda = \Gamma(\varphi)$ for some group homomorphism*

$$G \hookleftarrow H \xrightarrow{\varphi} \Sigma_n.$$

□

Definition 4.2.7. We call such subgroups *graph subgroups*, and, as it is clear that these form a family, denote the family of all graph subgroups of $G \times \Sigma_n$ by $\Gamma(G, \Sigma_n)$.

Thus, "Σ_n-free and G-contractible" does *not* specify the homotopy type of $\mathcal{O}(n)^\Gamma$ for non-trivial graph subgroups Γ, and our example above is the model with the $G \times \Sigma_n$-action as free as possible, with all the extra fixed-point spaces empty. Dually, we have the following:

Definition 4.2.8 ([CW91])**.** A *G-E_∞-operad* is an operad in G-spaces such that, for each n,

$$\mathcal{O}(n)^\Lambda \sim \begin{cases} * & \Lambda = \Gamma(\varphi) \text{ is a graph subgroup of } G \times \Sigma_n \\ \varnothing & \text{otherwise} \end{cases}$$

Equivalently, $\mathcal{O}(n) = E_G\Sigma_n$ is a universal space for G-equivariant Σ_n-bundles.

Algebras over these have the rich structure we desire:

Theorem 4.2.9 ([CW91, Theorem 1]). *If X is a group-like G-space of the homotopy type of a G-CW complex, which further has an action of a G-E_∞-operad, then X is G-homotopy equivalent to an equivariant infinite loop space.* $\square$

The upshot of this discussion is that the model structure on topological G-operads induced by the technology of Stephan on the orbit category is *insufficient*, as it is too coarse to detect the differences between G-trivial E_∞-operads encoding infinite loop spaces with G-actions and genuine G-E_∞-operads encoding equivariant infinite loop spaces.

Thus, in order to talk about the homotopy theory of G-operads, we must be able to detect the fixed-point information for all graph subgroups.

Definition 4.2.10. We say $\mathcal{V}\mathsf{Op}^G_{\{*\}}$ admits the *genuine equivariant* model structure if there exists a model structure where $f : \mathcal{O} \to \mathcal{P}$ is a weak equivalence (respectively, fibration) if $f(n)^\Gamma : \mathcal{O}(n)^\Gamma \to \mathcal{P}(n)^\Gamma$ is one in $\mathcal{V}$ for all graph subgroups $\Gamma \leq G \times \Sigma_n$ and all n.

A fundamental challenge to proving the existance of such a model structure is that said structure is no longer the lifting of a projective model structure on a diagram category: for each indexing element $n \in \mathbb{N}$, there is a *different* model structure we need to worry about. Thus, in particular, the machinery of [BM03] will also not provide us with an easy solution.

4.2.2 N_∞ Operads

While the original definition of equivariant E_∞-operad we gave did not sufficiently capture *all* of the equivariant complexities, precisely the idea that it is insufficient leads us to the following conclusion:

there are multiple notions of equivariant homotopy commutativity.

The two examples in the previous section provide two different versions. However, as we noted, there is an entire family worth of subgroups we can play around with. To that end, Blumberg-Hill made the following definition:

Definition 4.2.11 ([BH16])**.** An N_∞*-operad* is any G-operad of spaces such that $\mathcal{O}(n) \sim E\mathcal{F}_n(\mathcal{O})$, where $\mathcal{F}_n(\mathcal{O})$ is family of graph subgroups of $G \times \Sigma_n$ which contains all "trivial" graph subgroups of the form $H \times \{1\}$.

That is, $\mathcal{O}(n)^\Gamma \simeq *$ for all $\Gamma \in \mathcal{F}_n(\mathcal{O})$, and is empty otherwise. In particular, for all N_∞ operads, $\mathcal{O}(n)$ is Σ_n-free and G-contractible.

Lemma 4.2.12. *The underlying non-equivariant operad of any N_∞-operad is an E_∞-operad.*

Example 4.2.13. Examples include generalizations of non-equivariant E_∞-operads, namely the linear isometries, little disks, and Steiner operads for (not-necessarily-complete) G-universes U, where we recall a *G-universe* is a countably infinite-dimensional orthogonal G-representation which contains each finite-dimensional subrepresentation infinitely often and for which U^G is non-empty.

Definition 4.2.14. We say a map $f : \mathcal{O} \to \mathcal{P}$ of N_∞-operads is a *weak equivalence* if $f^\Gamma : \mathcal{O}(n)^\Gamma \to \mathcal{P}(n)^\Gamma$ is a weak equivalence of spaces for all graph subgroups $\Gamma \leq G \times \Sigma_n$. Denote by $\mathrm{Ho}(N_\infty)$ the homotopy category of N_∞-operads under weak equivalences.

Blumberg-Hill characterized N_∞-operads up to weak equivalence, by studying the required compatibilities between the various families $\mathcal{F}_n$ in the defining structure of some N_∞-operad $\mathcal{O}$. This led to the notion of an indexing system.

Definition 4.2.15. A *symmetric monoidal coefficient system* is a contravariant functor $\underline{\mathcal{C}} : O_G^{op} \to \mathsf{Cat}_{\mathsf{Sym}}$, the category of symmetric monoidal categories and strong symmetric monoidal functors.

Example 4.2.16. The canonical example is the system $\underline{\mathsf{Set}}$, which sends G/H to $(\mathsf{Set}^H, \amalg)$.

We will be interested in specific sub-systems of $\underline{\mathsf{Set}}$.

Definition 4.2.17. An *indexing system* is a full symmetric monoidal sub-coefficient system $\underline{\mathcal{F}}$ of $\underline{\mathsf{Set}}$ that contains all trivial H-sets for each $H \leq G$ and is closed under

(1) finite limits and

(2) "self-induction": if $H/K \in \underline{\mathcal{F}}(G/H)$ and $A \in \underline{\mathcal{F}}(G/K)$, then $H \cdot_K A \in \underline{\mathcal{F}}(G/H)$.

Let $\mathbb{I}$ denote the poset category of indexing systems under inclusion.

In particular, we note that any indexing system $\mathcal{F}$ is closed under isomorphisms, restrictions, subobjects, products, coproducts, and self-induction.

Any subsystem $\underline{\mathcal{F}} \subseteq \underline{\mathsf{Set}}$ induces a collection $\mathcal{F} = \{\mathcal{F}_n\}$ of families of graph subgroups of $G \times \Sigma_n$, and vice versa.

Definition 4.2.18. Given $\underline{\mathcal{F}} \subseteq \underline{\mathsf{Set}}$, we say $\Gamma(\varphi) \leq G \times \Sigma_n$ is $\mathcal{F}$-*admissible* if $\{1, 2, \ldots, n\}$ with the H-set structure induced by φ is in $\underline{\mathcal{F}}(G/H)$.

Conversely, given a collection $\mathcal{F} = \{\mathcal{F}_n\}$ of graph subgroups, we say an H-set A of cardinality n is $\mathcal{F}$-*admissible* if, for any choice of bijection $A \to |A|$, the map $\varphi : H \to \Sigma_{|A|}$ defining the H-set structure on A, we have $\Gamma(\varphi) \in \mathcal{F}_n$. We will abuse notation and denote by Γ_A any graph subgroup $\Gamma(\varphi)$ constructed as above (though we note that Γ_A is defined up to conjugation by elements in $\Sigma_{|A|}$).

Lemma 4.2.19. *These are inverse operations.* □

Definition 4.2.20. We call a collection $\mathcal{F} = \{\mathcal{F}_n\}$ *indexing* if $\underline{\mathcal{F}}$ is an indexing system.

By definition, we have a collection $\mathcal{F}(\mathcal{O}) = \{\mathcal{F}_n(\mathcal{O})\}$ associated to any N_∞-operad \$$O$, and hence a subsystem $\underline{\mathcal{F}}(\mathcal{O})$ of $\underline{\mathsf{Set}}$.

Theorem 4.2.21 ([BH16, Corollary 5.6])**.** *For any N_∞-operad $\mathcal{O}$, the associated coefficient system $\underline{\mathcal{F}}(\mathcal{O})$ is an indexing system. Moreover, $\underline{\mathcal{F}}$ defines a fully-faithful functor* $\mathrm{Ho}(N_\infty) \to \mathbb{I}$.

Blumberg-Hill also make the following conjecture:

Conjecture 4.2.22 ([BH16])**.** *The natural functor* $\mathrm{Ho}(N_\infty) \to \mathcal{I}$ *is essentially surjective, and hence an equivalence of categories.*

This conjecture will be confirmed below in Corollary 4.3.11.

Remark 4.2.23. Gutierrez-White [GW] have independently announced a confirmation of this conjecture.

Remark 4.2.24. Blumberg-Hill also offer a candidate for a more categorical model for these N_∞-operads, based on the *equivariant Barrett-Eccles operad* of [GMM12]. However, the author has shown that these candidate objects fail to have the desired structure for almost any group or any indexing system; see Appendix B for more details.

We note that the structure theorem says that $\mathcal{F}(\mathcal{O})$-admissible sets are closed under restrictions, conjugations, isomorphisms, products, coproducts, subobjects, and self-induction.

Remark 4.2.25. The algebra of indexing systems has also been studied by Blumberg-Hill [BH16]. In particular, they have shown that the poset of indexing systems is isomorphic to the poset of subcategories $\mathcal{D}$ of F^G such that the category of restricted polynomials of G-sets

$$X \leftarrow A \xrightarrow{f} B \rightarrow Y$$

with $f \in \mathcal{D}$ is a subcategory of the full category of polynomials.

4.2.3 Evaluation on G-Sets

Given a single-coloured G-operad $\mathcal{O}$, one method to access the desired fixed-point data needed for the genuine equivariant model structure (cf. 4.2.10) more explicitly is to change the underlying category of symmetric sequences, in particular so our operad is evaluated on all finite G-sets (and finite H-sets for $H \leq G$). Specifically, given a symmetric G-sequence

$X \in (\mathcal{V}^G)^{\Sigma^{op}}$, consider the left Kan extension induced by the span

$$\begin{tikzcd}
\Sigma \arrow[r, "X"] \arrow[d, hook, "i"'] \arrow[dr, Rightarrow] & \mathcal{V}^G \arrow[r, hook] & \underline{\mathcal{V}}^G \\
\mathsf{F}^G \arrow[d, hook] & & \\
\underline{\mathsf{F}}^G \arrow[uurr, "\mathrm{Lan}=i_!X"'] & &
\end{tikzcd}$$

where we recall $\underline{\mathcal{C}}^G$ is the G-set-enriched category of G-objects in $\mathcal{C}$. Explicitly, we see that

$$i_!X(A) \simeq X(|A|) \times_{\Sigma_{|A|}} \mathrm{Iso}(n, A).$$

Remark 4.2.26. We make a couple of observations:

(1) By first forgetting to an H-operad, this construction yields evaluations on all H-sets for all $H \leq G$.

(2) If A is an H-set of cardinality n, then $X(A)$ is non-equivariantly isomorphic to $X(n)$. However, the G-action on A is "twisting" the Σ_n-action, and in fact $X(A) \in \mathcal{V}^{G \ltimes \mathrm{Iso}(A,A)}$.

(3) For the same A, we note that $X(A)^H \simeq X(n)^{\Gamma_A}$ where $\Gamma_A = \Gamma(\varphi)$ for some (any) homomorphism

$$G \hookleftarrow H \xrightarrow{\varphi} \Sigma_n$$

which encodes the H-set structure on A.

Moreover, it is not hard to check that $i_!X$ is in fact an enriched functor, nor that the following lemma holds.

Lemma 4.2.27. *The left Kan extension over i induces an adjoint pair*

$$i_! : \mathsf{Sym}^G \leftrightarrows \mathrm{Fun}^G(\underline{\mathsf{F}}^G, \underline{\mathcal{V}}^G) : i^*$$

where $\mathrm{Fun}^G(-,-)$ *is the category of* G*-enriched functors. Moreover, this adjunction is in fact an equivalence of categories.* $\square$

Previous work of the author [Bon16] has shown that these evaluations on G-sets can be incorporated into an operadic structure, yielding composition structure maps

$$X(A) \times \prod_{a \in A} X(B_a) \to X(\amalg_a B_a)$$

for any G-sets A and B_a, which are associative, unital, natural in the A and (B_a), and "almost equivariant". This last property refers to the fact that we are taking a G-set-indexed product over a collection of G-sets with no known compatibility, and thus we cannot "act on the indices" as we would like to; this would require the B_a to be $\mathrm{Stab}_G(a)$-sets, and compatibly so. This is one of the major flaws of the model of equivariant operads presented thus far: structure maps of the form, for example,

$$\mathcal{O}(H/K)^H \times \left(\prod_{H/K} \mathcal{O}(h^*K/L)^K \right)^H \to \mathcal{O}(H/L)^H \tag{4.1}$$

where $h^*K/L = hKh^{-1}/hLh^{-1}$, cannot be easily seen or analyzed effectively.

This will be one of the motivations to develop "genuine equivariant operads", as we do in Chapter 6.

4.2.4 Equivariantly Coloured Operads and another Free Operad Monad

We now analyze the situation where our G-set $\mathfrak{C}$ of colours need not be trivial. Thus, in addition to the structure found in Definition (cf. 2.2.15), we have an action of G on set of signatures, and the composition structure maps need to be natural over this action.

We can encode this categorically. As before (cf. Section 2.2.2), let $\Sigma_{/\mathfrak{C}}$ denote the set of $(n+1)$-tuples $\xi = (c_1, \ldots, c_n; c) \in \mathfrak{C}^{\times n} \times \mathfrak{C}$. However, unlike in the non-equivariant case, $\Sigma_{/\mathfrak{C}}$ now has a G-action which is not currently being recorded. To that end, we define the following Grothendieck category:

Definition 4.2.28. Let $\Sigma_{/\mathfrak{C}} = \int_{G\times\Sigma} \mathsf{Sig}(\mathfrak{C})$ denote the Grothendieck construction associated to the functor

$$G \times \Sigma \xrightarrow{\mathsf{Sig}(\mathfrak{C})} \mathsf{Set} \hookrightarrow \mathsf{Cat}$$

defined by $(*, n) \mapsto \mathfrak{C}^{\times n} \times \mathfrak{C}$, where Σ_n acts on $\mathfrak{C}^{\times n}$ by permutations, and G acts on $\mathfrak{C}^{\times n+1}$ diagonally.

Explicitly, objects of $\Sigma_{/\mathfrak{C}}$ are signatures, or $(n+1)$-tuples $\xi = (c_1, \dots, c_n; c_0)$ of colours, and maps $\xi \to \xi'$ are pairs $(g, \sigma) \in G \times \Sigma_n$ such that $c_i = gc'_{\sigma^{-1}i}$ for all $i \in \{0, 1, \dots, n\}$ (where σ is the identity on 0).

A $\mathfrak{C}$-*symmetric sequence* is a functor $X : \Sigma_{/\mathfrak{C}} \to \mathsf{Set}$.

For any $\mathfrak{C}$-symmetric sequence X and signature $\xi \in \mathfrak{C}^{\times n} \times \mathfrak{C}$, we observe that $X(\xi) \in \mathcal{V}^{\mathrm{Stab}_{G\times\Sigma_n}(\xi)}$.

Definition 4.2.29. A $\mathfrak{C}$-*coloured operad* is a $\mathfrak{C}$-symmetric sequence $\mathcal{O}$ with units $1_c \in \mathcal{O}(c; c)$ for each $c \in \mathfrak{C}$, and composition maps

$$\mathcal{O}(a_1, \dots, a_n; a_0) \times \prod_i \mathcal{O}(b_i^1, \dots, b_i^{k_i}; a_i) \to \mathcal{O}(b_1^1, \dots, b_n^{k_n}; a_0)$$

which are associative, G- and $\mathfrak{C}$-equivariant, and unital.

We denote by $\mathcal{V}\mathsf{Op}^G_{\mathfrak{C}}$ the category of $\mathfrak{C}$-coloured operads and maps of sequences which preserve the given structure.

Definition 4.2.30. The category $\mathcal{V}\mathsf{Op}^G$ is the Grothendieck construction on the functor $\mathsf{F}^G \to \mathsf{Cat}$ sending $\mathfrak{C}$ to $\mathcal{V}\mathsf{Op}^G_{\mathfrak{C}}$. Explicitly, maps $\mathcal{O} \to \mathcal{P}$ are given by G-maps of colours $f : \mathfrak{C}(\mathcal{O}) \to \mathfrak{C}(\mathcal{P})$ and a map $\mathcal{O} \to f^*\mathcal{P}$ in $\mathcal{V}\mathsf{Op}^G_{\mathfrak{C}}$.

Remark 4.2.31. Operads in the category $\mathcal{V}^G\mathsf{Op}$ do *not* have a G-action on their sets of objects; instead, for each signature $\xi \in \mathfrak{C}^{\times n} \times \mathfrak{C}$, we have $\mathcal{O}(\xi) \in \mathcal{V}^{G\times\mathrm{Stab}_{\Sigma_n}(\xi)}$.

We observe that there is a natural forgetful functor $\mathcal{V}\mathsf{Op}^G_{\mathfrak{C}} \to \mathcal{V}^{\Sigma/C}$. In fact, more is true:

Theorem 4.2.32. *The forgetful functor is part of a monadic adjunction*

$$\mathcal{V}\mathsf{Op}^G_{\mathfrak{C}} \underset{\mathsf{fgt}}{\overset{F}{\rightleftarrows}} \mathcal{V}^{\Sigma_{/\mathfrak{C}}}. \tag{4.2}$$

Remark 4.2.33. If $\mathfrak{C} = \{*\}$, then this is precisely the adjunction (3.1) for $\mathcal{V}^G$.

We observe that this agrees with our notion of morphism of $\mathfrak{C}$-coloured operads above, as maps of sequences which preserve the structure.

We prove this theorem by a series of lemmas, generalizing the methods of Section 3.2 to build the left adjoint directly.

Definition 4.2.34. Let $\mathbb{T}_{0/\mathfrak{C}}$ be the category of "$\mathfrak{C}$-coloured planar trees" defined as the Grothendieck of the following functor:

$$\begin{aligned} \mathbb{T}_0 \times G &\longrightarrow Cat \\ (T, *) &\longmapsto B_{\mathfrak{C}} G^{E(T)} \end{aligned}$$

Explicitly, objects are pairs $(T, \mathfrak{c})$ with $T \in \mathbb{T}_0$ a planar tree, and $\mathfrak{c} : E(T) \to \mathfrak{C}$ is a *colouring* of the edges. Morphisms are given by pairs

$$(\varphi, g) : (T, \mathfrak{c}) \to (T', g_* \varphi^* \mathfrak{c})$$

where $\varphi : T \to T'$ is non-planar isomorphism of trees in $\mathbb{T}_0$, $g \in G$, and $g_*\varphi^*\mathfrak{c}$ is the composite

$$g_*\varphi^*\mathfrak{c} : E(T') \xrightarrow{\varphi^{-1}} E(T) \xrightarrow{\mathfrak{c}} \mathfrak{C} \xrightarrow{g} \mathfrak{C}.$$

Relations are just given by composition in $\mathbb{T}_0$ and G: $(\varphi, g) \circ (\psi, h) = (\varphi\psi, gh)$.

Lemma 4.2.35. $\Sigma_{/\mathfrak{C}}$ *is isomorphic to the subcategory of* $\mathbb{T}_{0/\mathfrak{C}}$ *spanned by corollas. In particular, the "valence" functor extends to a map* $\mathrm{val} : \mathbb{T}_{0/\mathfrak{C}} \to \Sigma_{/\mathfrak{C}}$. $\square$

More, the vertex functor (and hence the nerve-evaluation maps) also extend:

Definition 4.2.36. Define $\mathbb{V} : \mathbb{T}_{0/\mathfrak{C}} \to \mathsf{F}_0 \wr \Sigma_{/\mathfrak{C}}$ by

$$(T, \mathfrak{c}) \mapsto (V(T), v \mapsto (T_v, \mathfrak{c}|_{T_v})).$$

Definition 4.2.37 (cf. Definition 3.2.3)**.** Define the endofunctor $\mathbb{F}_{/\mathfrak{C}}$ on $\mathcal{V}^{\Sigma_{/\mathfrak{C}}}$ by sending X to the left Kan extension

$$\begin{tikzcd}
\mathbb{T}_{0/\mathfrak{C}} \arrow[r, "\mathbb{V}"] \arrow[d, "val"] \arrow[drr, phantom, "\Rightarrow"] & \mathsf{F}_0 \wr \Sigma_{/\mathfrak{C}} \arrow[r, "X"] & \mathsf{F}_0 \wr \mathcal{V}^{op} \arrow[r, "\otimes"] & \mathcal{V} \\
\Sigma_{/\mathfrak{C}} \arrow[urrr, dotted, "\mathrm{Lan} := \mathbb{F}_{/\mathfrak{C}} X"'] & & &
\end{tikzcd}$$

As before, we can present iterates of this endofunctor by considering certain left Kan extensions over pullbacks.

Definition 4.2.38 (cf. Definition 3.1.11)**.** Define $\mathbb{T}_{1/\mathfrak{C}}$ to be the pullback in categories

$$\begin{tikzcd}
\mathbb{T}_{1/\mathfrak{C}} \arrow[r, "\mathbb{V}"] \arrow[d, "d_1"] & \mathsf{F}_0 \wr \mathbb{T}_{0/\mathfrak{C}} \arrow[d, "val"] \\
\mathbb{T}_{0/\mathfrak{C}} \arrow[r, "\mathbb{V}"] & \mathsf{F}_0 \wr \Sigma_{/\mathfrak{C}}
\end{tikzcd}$$

The proof of the following is completely analogous to the proof of Lemma 3.2.6.

Lemma 4.2.39. *Given $X \in \mathcal{V}^{\Sigma_{/\mathfrak{C}}}$, the sequence $\mathbb{F}_{/\mathfrak{C}}\mathbb{F}_{/\mathfrak{C}}X$ is isomorphic to the left Kan extension*

$$\mathbb{F}_{/\mathfrak{C}}\mathbb{F}_{/\mathfrak{C}}X \simeq \mathrm{Lan}_{val \circ d_1}(\otimes \circ \mathsf{F}_0 \wr N_X \circ \mathbb{V}).$$

□

Moreover, we also have the assembly map $d_0 : \mathbb{T}_{0,/\mathfrak{C}} \to \mathbb{T}_{0/C}$, as the colours and equivariance are coherent via the pullback construction.

Now, the statements in Section 3.2 and the proofs in Section 6.3 completely generalize to this small increase in complexity, yielding the following:

Proposition 4.2.40 (cf. Proposition 3.2.7)**.** $\mathbb{F}_{/\mathfrak{C}}$ *is a monad on the category* $\mathcal{V}^{\Sigma_{/\mathfrak{C}}}$. □

Finally, the following result, that this monad precisely describes objects in $\mathcal{V}\mathsf{Op}_{\mathfrak{C}}^G$, completes our proof of Theorem 4.2.32.

Proposition 4.2.41. *The categories $\mathcal{V}\mathsf{Op}_{\mathfrak{C}}^G$ and $\mathbb{F}_{/\mathfrak{C}}$-algebras in $\mathcal{V}^{\Sigma_{/\mathfrak{C}}}$ are equivalent, and $\mathbb{F}_{/\mathfrak{C}} = F$ in the adjunction from (4.2).*

Proof. Given an $\mathbb{F}_{/\mathfrak{C}}$-algebra X, we define our composition structure maps γ by noting that the domain of said maps

$$X(a_1, \dots, a_n; a_0) \otimes X(b_1^1, \dots, b_1^{k_1}; a_1) \otimes \dots \otimes X(b_n^1, \dots, b_n^{k_n}; a_n)$$

has a natural map into $\mathbb{F}_{/\mathfrak{C}} X(b_1^{k_1}, \dots, b_n^{k_n}; a_0)$ along the inclusion given by the object $T = C_n \circ (C_{k_i})$ in $\mathbb{T}_{0/\mathfrak{C}}$, with the obvious edge colourings. Composing this with the algebra map of X provides us our structure map γ. We note that this is associative by the naturality with assembly data. Similarly, the unit corresponds to $c \in \mathfrak{C}$ is produced by assembling the c-coloured stick. Finally, the appropriate action of $G \times \Sigma_n$ on $\{X(\xi)\}_{|\xi|=n+1}$ is given by the functoriality of val with respect to those morphisms in $\Sigma_{/\mathfrak{C}}$ and $\mathbb{T}_{0/\mathfrak{C}}$.

Conversely, any $\mathfrak{C}$-coloured operad $\mathcal{O}$ has a natural $\mathbb{F}_{/\mathfrak{C}}$-algebra structure via iterated composition structure maps and the inclusion of the unit. □

Remark 4.2.42. At this point, it is worth noting that the above construction of $\mathbb{F}$ seems to be fairly general: given a span $\mathcal{D} \leftarrow \mathcal{C} \rightarrow \mathsf{F} \wr \mathcal{D}$, there should be necessary conditions so that we can build an $\mathbb{F}$-like monad on sequences $\mathcal{V}^{\mathcal{D}}$. We expect to return to analyzing the details of this generality in the future.

4.3 Conjectured and Semi Model Structures

We would like to use Theorem 4.2.32 to endow the category $\mathcal{V}\mathsf{Op}_{\mathfrak{C}}^G$ (and ultimately $\mathcal{V}\mathsf{Op}^G$) with a Quillen model structure that captures the interesting homotopical information discussed in Section 4.2.2. While we are ultimately unsuccessful (so far), we have successfully used it endow $\mathcal{V}\mathsf{Op}^G_{\{*\}}$ with a *semi*-model structure, which is sufficient, for example, to confirm Conjecture 4.2.22.

To that end, we fix a collection $\mathcal{F} = \{\mathcal{F}_n\}$ of families of graph subgroups of $G \times \Sigma_n$.

Definition 4.3.1. For a G-set $\mathfrak{C}$ and a signature $\xi \in \mathfrak{C}^n \times \mathfrak{C}$, define $\mathcal{F}_\xi$ to be the family $\mathcal{F}_n \cap \mathrm{Stab}_{G\times\Sigma_n}(\xi)$.

We identify the following maps in $\mathcal{V}\mathsf{Op}^G_{\mathfrak{C}}$.

Definition 4.3.2. A map $f : \mathcal{O} \to \mathcal{P}$ in $\mathcal{V}\mathsf{Op}^G_{\mathfrak{C}}$ is called a

- *$\mathcal{F}$-fibration* (resp. $\mathcal{F}$-weak equivalence) if for all n and all signatures $\xi \in \mathfrak{C}^{\times n} \times \mathfrak{C}$, $f(\xi) : \mathcal{O}(\xi) \to \mathcal{P}(\xi)$ is one in $\mathcal{V}^{\mathrm{Stab}(\xi)}_{\mathcal{F}_\xi}$; that is, $f(\xi)^\Gamma$ is a fibration (resp weak equivalence) in $\mathcal{V}$ for all $\Gamma \in \mathcal{F}_n \cap \mathrm{Stab}(\xi)$.
- *$\mathcal{F}$-cofibration* if it has the left lifting property against all map which are both $\mathcal{F}$-fibrations and $\mathcal{F}$-weak equivalences.
- *$\mathcal{F}$-level-cofibration* if for all n and all signatures ξ in $\mathfrak{C}^{\times n} \times \mathfrak{C}$, $f(\xi)$ is a cofibration in $\mathcal{V}^{\mathrm{Stab}\,\xi}_{\mathcal{F}_\xi}$.

In particular, $\mathcal{O}$ is *$\mathcal{F}$-cofibrant* if $\varnothing \to \mathcal{O}$ is an $\mathcal{F}$-cofibration, where $\varnothing$ is the initial object of $\mathcal{V}$.

Definition 4.3.3. The *$\mathcal{F}$-model structure* is the unique model structure on $\mathcal{V}\mathsf{Op}^G_{\mathfrak{C}}$, if it exists, with (co)fibrations and weak equivalences as just defined above. If it does, we say that the pair $(\mathcal{V}, \mathfrak{C})$ is *$\mathcal{F}$-admissible.*

The following is expected to be true.

Conjecture 4.3.4. *For suitable $\mathcal{V}$ (including* sSet*,* Top*,* $\mathsf{Ch}(R)$*, and* Sp*), $(\mathcal{V}, \mathfrak{C})$ is $\mathcal{F}$-admissible for any G-set $\mathfrak{C}$.*

We expect to be able to adapt our proof of Theorem 5.3.5 below to prove this for "suitable" categories $\mathcal{V}$. Any choice of "suitable" will necessarily include at least the following properties:

Definition 4.3.5. We say $\mathcal{V}$ satisfies ASSUMPTION 1 if the following hold:

(1) $\mathcal{V}$ is a cofibrantly-generated closed symmetric monoidal model category, and

(2) for all finite groups G and all families $\mathcal{F}_n$ of graph subgroups of $G \times \Sigma_n$, $\mathcal{V}^{G\times\Sigma_n}$ is $\mathcal{F}$-cellular (4.1.14).

Let us now repackage the $\mathcal{F}$-model structure so that the relevance of the adjunction (4.2) becomes evident. If $\mathfrak{C} = \{*\}$, this is immediate by definition, as $\mathcal{V}^{\Sigma_{/\mathfrak{C}}} = \prod_n \mathcal{V}^{G\times\Sigma_n}$, and we can just endow each $\mathcal{V}^{G\times\Sigma_n}$ with the $\mathcal{F}_n$-model structure via Theorem 4.1.15. For general $\mathfrak{C}$, we need to ensure we can endow $\mathcal{V}^{\Sigma_{/\mathfrak{C}}}$ with an appropriate $\mathcal{F}$-model structure.

To that end, we first prove an equivariant strengthening of Theorem 1.2.8:

Theorem 4.3.6. *Suppose $\mathcal{V}$ satisfies* ASSUMPTION 1*, and $\mathcal{D}$ is any small category such that for all $d \in \mathcal{D}$, the set of endomorphisms $\mathcal{D}(d,d)$ is in fact a group Π_d. Further, suppose we are given families $\mathcal{F}_d \subseteq \mathcal{L}(\Pi_d)$ for all $d \in \mathcal{D}$. Then the diagram category $\mathcal{V}^{\mathcal{D}}$ has a cofibrantly-generated projective model structure, where $f : X \to Y$ is a weak equivalence (respectively, fibration) if $f(d) : X(d) \to Y(d)$ is so in $\mathcal{V}^{\Pi_d}_{\mathcal{F}_d}$.*

Moreover, if $X \in \mathcal{V}^{\mathcal{D}}$ is cofibrant, it is also cofibrant levelwise.

Proof. This is a straightforward synthesis of [Hir03, 11.6.1] with the technology of Stephan [Ste16]. Let I and J be the generating (trivial) cofibrations of $\mathcal{V}$, we have

$$I_{\mathcal{F}_d} = \{\Pi_d/H \cdot i \mid i \in I,\ H \in \mathcal{F}_d\}$$

and similarly $J_{\mathcal{F}_d}$ are generating cofibrations for $\mathcal{V}^{\Pi_d}_{\mathcal{F}_d}$. Similarly, the product category $\mathcal{V}^{\mathrm{Ob}(\mathcal{D})}$ has a cofibrantly-generated levelwise model structure built by [Ste16, Theorem A.1] along the adjoints

$$\mathcal{V}^{\mathrm{Ob}\mathcal{D}} \underset{ev_d}{\overset{(-)_d}{\rightleftarrows}} \mathcal{V}^{\Pi_d}_{\mathcal{F}_d}$$

sending V to $V_d = \delta_d(-) \cdot V$, where δ_d evaluates to the tensor unit I on d, and the initial object of $\mathcal{V}$ otherwise. Thus we have generating cofibrations

$$I_{\mathcal{F}}^{\mathrm{Ob}\mathcal{D}} = \{\delta_d(-) \cdot \Pi_d/H \cdot i \mid d \in D,\ i \in I,\ H \in \mathcal{F}_d\}\,,$$

and similarly for trivial cofibrations.

Lastly, we have the forgetful adjunction

$$\mathcal{V}^{\mathcal{D}} \underset{\mathsf{fgt}}{\overset{F}{\leftrightarrows}} \prod_{d\in\mathcal{D}} \mathcal{V}_{\mathcal{F}_d}^{\Pi_d}$$

where F sends X to the diagram

$$FX(d) = \coprod_{c\in\mathcal{D}} X(c) \cdot_{\Pi_c} \mathcal{D}(c,d).$$

Define our set of trivial cofibrations in $\mathcal{V}^{\mathcal{D}}$ by $I_{\mathcal{F}}^{\mathcal{D}} := F(I_{\mathcal{F}}^{\mathrm{Ob}\mathcal{D}})$, and we observe

$$I_{\mathcal{F}}^{\mathcal{D}} = \{\mathcal{D}(d,-) \cdot_{\Pi_d} \Pi_d/H \cdot i \mid d \in \mathcal{D}, i \in I,\ H \in \mathcal{F}_d\}\,;$$

similarly define $J_{\mathcal{F}}^{\mathcal{D}}$. To confirm that this is indeed a model structure, using the Transfer Principle 1.2.6, it suffices to check that pushouts of the form

$$\begin{tikzcd} FA \arrow[r] \arrow[d, "F_c i"] & X \arrow[d] \\ FB \arrow[r] & Y \end{tikzcd}$$

with $\iota : A \to B$ in $\prod_d \mathcal{V}_{\mathcal{F}_d}^{\Pi_d}$, $c \in \mathcal{D}$, and $X \in \mathcal{V}^{\mathcal{D}}$, if i is a generating (trivial) cofibration, then the pushout $X \to Y$ is an underling (trivial) cofibration in $\prod_d \mathcal{V}_{\mathcal{F}_d}^{\Pi_d}$. Indeed, $F\iota$ is of the form

$$\mathcal{D}(d,-) \cdot_{\Pi_d} \Pi_d/H \cdot i$$

for some $i \in I$ (or $j \in J$), and since pushouts in $\mathcal{V}^{\mathcal{D}}$ are computed levelwise, at any c the left hand side is just a coproduct (over $\mathcal{D}(d,c)$) of generating (trivial) cofibrations in $\mathcal{V}_{\mathcal{F}_d}^{\Pi_d}$, and hence the pushout is one as well.

The moreover is immediate from the discussion of the pushout above. □

We observe that $\Sigma_{/\mathfrak{C}}(\xi, \xi)$ is precisely $\mathrm{Stab}_{G\times\Sigma_n}(\xi)$ (where ξ is a signature in $\mathfrak{C}^{\times n} \times \mathfrak{C}$). Thus, the following is a special case of the above:

Corollary 4.3.7. *Suppose $\mathcal{V}$ satisfies* Assumption 1*. Then, for any G-set $\mathfrak{C}$, and any collection $\mathcal{F} = \{\mathcal{F}_n\}$ of graph subgroups, the diagram category $\mathcal{V}^{\Sigma_{/\mathfrak{C}}}$ has the projective $\mathcal{F}$-model structure, where $f : X \to Y$ is a weak equivalence (respectively, fibration) if $f(\xi) : X(\xi) \to Y(\xi)$ is so in $\mathcal{V}_{\mathcal{F}_n}^{\mathrm{Stab}_{G\times\Sigma_n}(\xi)}$, for ξ a signature of length $n+1$ and $\mathcal{F}_\xi = \mathcal{F}_n \cap \mathrm{Stab}_{G\times\Sigma_n}(\xi)$.*

Moreover, if $X \in \mathcal{V}^{\mathcal{D}}$ is cofibrant, then it also is so levelwise. □

We denote this model category by $\mathcal{V}_{\mathcal{F}}^{\Sigma_{/\mathfrak{C}}}$. The following is then immediate:

Proposition 4.3.8. *The $\mathcal{F}$-model structure on $\mathcal{V}\mathsf{Op}_{\mathfrak{C}}^{G}$, if it exists, is precisely the transfered model structure from $\mathcal{V}_{\mathcal{F}}^{\Sigma_{/\mathfrak{C}}}$ via the adjunction (4.2).* □

Thus, as is often the case, the existence of the $\mathcal{F}$-model structure comes down to the Transfer Principle, and in particular showing that all maps in $J_{\mathcal{F}}^{\Sigma_{/\mathfrak{C}}}$-cell are weak equivalences.

4.3.1 Partial Results

We have partial results in the case where $\mathfrak{C} = *$ and $\mathcal{F} = \{\mathcal{F}_n\}$ is a "weak indexing system" (see 5.1.55). In particular, while we cannot say that the $\mathcal{F}$-model structure exists, we do have that the $\mathcal{F}$-*semi*-model structure exists:

Theorem 4.3.9. *Let $\mathcal{V}$ satisfy* Assumption 1*, and let $\mathcal{F}$ be a weak indexing system. Then $\mathcal{V}\mathsf{Op}_{\{*\}}^{G}$ can be endowed with the $\mathcal{F}$-semi-model structure, where the (co)fibrations are the $\mathcal{F}$-(co)fibrations, and the weak equivalences are the $\mathcal{F}$-weak equivalences. Moreover, $\mathcal{F}$-cofibrations with cofibrant domains are level $\mathcal{F}$-cofibrations.*

Proof. This is an immediate corollary of Theorem 5.3.3 and Corollary 5.3.5 from Section 5.3, by applying Theorem 1.2.10. □

We will delay the stating and proof of these two technical results until we have the terminology and technology of "$\mathcal{F}$-admissible trees".

Remark 4.3.10. In particular, the two cited results say that all maps in $J_{\mathcal{F}}^{\Sigma/\mathfrak{C}}$ with *cofibrant source* are $\mathcal{F}$-weak equivalences in $\mathcal{V}^{\Sigma/\mathfrak{C}}$ (as trivial cofibrations are preserved by transfinite compositions). Thus, if we could relax this condition, then the Transfer Principle 1.2.6 would imply that the above semi-model structure is in fact a Quillen model structure. We expect this will come from analyzing the proof of Theorem 5.3.3, and carefully noting what stronger conditions of $\mathcal{V}$ could replace the cofibrancy of $\mathcal{P}$.

While this structure is weaker than a true model structure, it is sufficient to prove the following:

Corollary 4.3.11. *For $\mathcal{V} = \mathsf{Top}$ and $\mathcal{F}$ any weak indexing system, there exists an operad $N\mathcal{F}$ such that $N\mathcal{F}(n)^{\Gamma} \simeq *$ if $\Gamma \in \mathcal{F}(n)$, and is empty otherwise. In particular,* $\mathrm{Ho}(N_\infty\text{-}\mathsf{Op}) \to \mathbb{I}$ *is an equivalence of categories, proving Conjecture 4.2.22.*

Proof. Recall that $\mathsf{Comm}(n) = *$ for all n. Consider the functorial factorization

$$\varnothing \to N\mathcal{F} \xrightarrow{\sim} \mathsf{Comm}$$

in $\mathcal{V}\mathsf{Op}^G$ with the $\mathcal{F}$-semi-model structure. Since the initial operad is cofibrant, Theorem 4.3.9 implies that $\varnothing \to N\mathcal{F}$ is a level $\mathcal{F}$-cofibration, and hence each $N\mathcal{F}(n)$ is cofibrant in $\mathsf{Top}_{\mathcal{F}_n}^{G\times\Sigma_n}$; hence, for all $\Gamma \notin \mathcal{F}_n$, $N\mathcal{F}(n)^{\Gamma} = \varnothing$. Further, since $N\mathcal{F}$ is $\mathcal{F}$-equivalent to $*$, $N\mathcal{F}(n)^{\Gamma} \simeq *$ for all $\Gamma \in \mathcal{F}_n$. Hence, each $N\mathcal{F}(n)$ is a universal space for $\mathcal{F}_n$, as desired. □

Chapter 5

Equivariant Dendroidal Sets

In the previous chapter, we saw the need for more subtle control over the combinatorics and equivariance of G-operads. Moreover, we saw that some of these subtleties were mediated by certain closure conditions on "indexing systems" of finite G-sets.

In this chapter, we construct an equivariant generalization of the dendroidal category, and show that presheaves on this category provide a diagrammatic interpretation of these closure constraints. This in turn will allow us to encode equivariant operads combinatorially. Lastly, we will mention the first homotopical comparison between the standard equivariant operads and these combinatorial models.

This chapter — specifically, the definition of G-trees and Ω_G, the categories $\mathsf{dSet}_{\mathcal{F}}$, the notions of equivariant face and horn maps, and the determination of the generating (trivial) cofibrations — is all joint with Luis Pereira; in fact, conversations leading to this work was the genesis of our current collaborative efforts on many fronts. The technical proof of the existence of the model structure (Theorem 5.2.23) is his alone.

5.1 Equivariant Trees

Again, we fix a finite group G.

In this section, we will introduce the new combinatorial object of an equivariant tree, or just "G-tree". Section 2.2 showed that trees encode operadic composition information; G-trees will additionally describe the G-equivariance of this structure. In particular, G-trees will be able to both

- record information about the fixed points on graph subgroups, and
- encode the combinatorics of compositions via grafting,

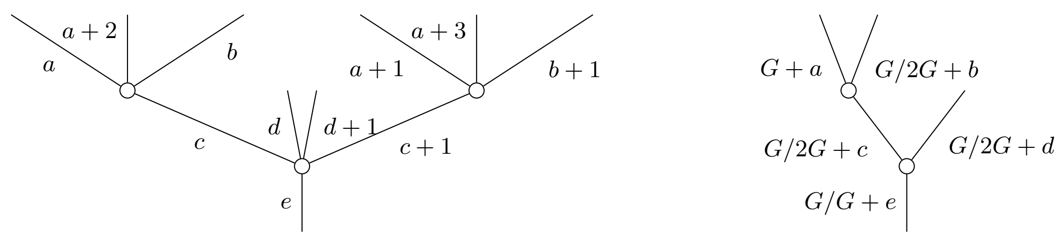
and do so in a way that these two pieces of structure interact coherently.

5.1.1 Examples

We will motivate the formal definition by first discussing some examples of equivariant trees.

A naïve first guess might be to consider "trees with G-action", or just Ω^G. These end up being necessary, but not sufficient, for our discussion. However, they are still of importance, and we will use them as our first examples.

Example 5.1.1. Let $G = \mathbb{Z}/4$. The following two diagrams provide representations of an element of Ω^G:



Representations of the kind on the left will be called the *expanded representation*. These are simply planar representations of the corresponding tree, where we have labeled the edges by names which reflect the G-action. In this example, $1 \in G$ acts by sending a to $a+1$, $a+1$ to $a+2$, b to $b+1$, etc. We highlight the fact that, implicitly, $a+4=a$, $b+2=b$, $c+2=c$, and $d+2=d$.

Representations of the kind on the right will be called the *orbital representation*. This reduced presentation is in fact the quotient of the expanded representation, by identifying edges in the same orbit. Importantly, we now label the edges by the orbit which has been collapsed.

Definition 5.1.2. Given a G-tree T, define $V_G(T) := V(T)/G$, where $V(T)$ is the G-set of vertices in the expanded representation. Equivalently, $V_G(T)$ is the set of vertices in the orbital representation. Given $G.v \in V_G(T)$, let $T_{G.v}$ denote the G-corolla "surrounding" $G.v$, namely the G-tree with elements

$$\coprod_{v \in G.v} t_v^{\uparrow} \amalg t_v$$

and generating relations $t_v^{\uparrow} \leq t_v$. In the orbital representation, $T_{G.v}$ is also equivalent to the corolla "surrounding" the node representing $G.v$.

The *degree* of a G-tree T is defined by $deg(T) = |V_G(T)|$.

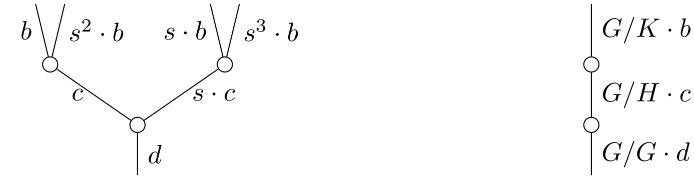
Remark 5.1.3. We note that the expanded representation of an object $T \in \Omega^G$ necessarily includes additional data not found in the category, namely

(1) a planar structure on each tree component (c.f. Section 2.3.4) and on the root G-set; and

(2) a choice of basepoint in each orbit of $E(T)$.

Note that (2) is only a naming convention in the expanded representation, but on the orbital representation affects the identification of the orbit of edges with a particular orbit in G. With that in mind, we record that the expanded notation is only unique up to non-planar isomorphism, and the orbital representation is only unique up to (edge-wise) isomorphism in the orbit category O_G.

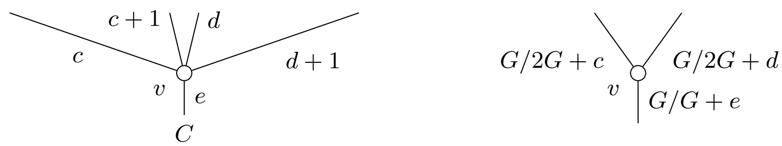
Example 5.1.4. Let $G = D_8 = \langle r, s \mid r^2 = s^4 = rsrs = 1\rangle$, and consider the subgroups $H = \langle r, s^2\rangle$ and $K = \langle r\rangle$. We again give two representation of the G-tree below.



(5.1)

We note that, as above, we have chosen representatives of the cosets in G/H and G/K, i.e. we could have written $\{c, rs \cdot c\}$ as opposed to $\{c, s \cdot c\}$ — as before, we are making the implicit assumption that $s \cdot c = rs \cdot c$.

Now, let's consider the root corolla C from Example 5.1.1:



(5.2)

At this point, it is natural to ask:

What operadic information is encoded by this tree with G-action?

That is, which operations φ of an operad $\mathcal{P}$ can live at the node v? We note that the input edges of v form a G-set $A = \{b, b+1, c, c+1\} \simeq G/2G \amalg G/2G$, and so φ should be in $\mathcal{P}(A)$ (using the notation from Section 4.2.3). However, there is another restriction: the operation in $\Omega(C)$ generated by v is fixed by G, and hence we have $\varphi \in \mathcal{P}(A)^G$.

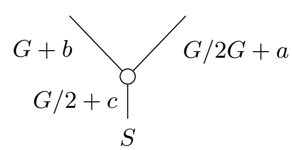
Equivalently, if we pick an (arbitrary) planarization of C, say the one given implicitly by our depiction above, and consider φ as an element of $\mathcal{P}(4)$, we see that this planar tree is preserved by the graph of the homomorphism $\alpha : G \to \Sigma_4$, $1 \mapsto (12)(34)$, and hence we must have $\varphi \in \mathcal{P}(4)^{\Gamma(\alpha)}$. Note that by Observation 4.2.26, $\mathcal{P}(4)^{\Gamma(\alpha)} \simeq \mathcal{P}(A)^G$.

Remark 5.1.5. We observe and record that the homomorphism α defines a G-action on the set $\{1, 2, 3, 4\}$ equivalent to $G/2G \amalg G/2G$, which is suggested by the orbital notation. In fact, this relationship between the orbital notation and the represented G-set will always be true, and is one of the motivating features of our definition.

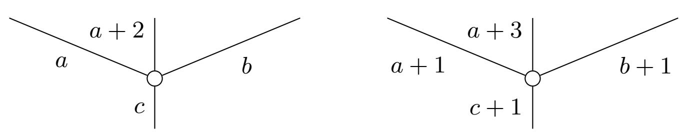
The above examples allow us to access graph fixed points for G-sets. However, as we saw in Chapter 4, in order to encode all the interesting G-homotopy information about G-operads, we need a way to access graph fixed points for *all* H-sets for any subgroup $H \leq G$.

Additionally, as we mentioned earlier, trees are only useful in encoding operadic information because there is a way to graft them together. However, there is currently no rule to combine two trees with G-action into a third, and in fact in general there *cannot* be, as there is no way to ensure the resulting object will have a G-action that reflects the original ones. The only way to guarantee that the resultant of grafting has a well-defined inherited G-action is to allow ourselves to graft trees with *smaller* group actions, and further to be able to graft *multiple* copies at once in order to cover an entire orbit's worth of leaf edges simultaneously. Thus, it makes sense for us to also consider "trees" of the from $G \cdot_H T_H$ for any tree with H-action $T_H \in \Omega^H$, and to allow ourselves to graft $G \cdot_H T_H$ to a leaf orbit isomorphic to G/H.

Example 5.1.6. $G = \mathbb{Z}/6$. The equivariant tree S with orbital representation



"should" encode operations $\varphi \in \mathcal{P}(3)^{\Gamma_{2G\amalg 2G/2G}}$. Indeed, the above tree S has expanded representation



and hence encodes $\varphi \in \mathcal{P}(3)^{\Gamma(\beta)}$, for $\beta : G/2G \to \Sigma_3$ sending $1 \mapsto (12)(3)$; however, it also encodes the entire orbit of φ, namely $\varphi + 1 \in \mathcal{P}(3)^{\Gamma(\beta+1)}$.

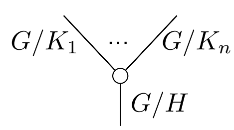
Remark 5.1.7 (A Remark on Grafting of Subtrees)**.** We note that the tree from Example 5.1.6 is a G-equivariant sub-broad poset of the tree in Example 5.1.1, and in fact this will be a *subtree* once we establish our formal definitions. Moreover, we may equivariantly *graft* the tree from Example 5.1.6 onto the root corolla from Diagram (5.2) — via gluing together edges with the same names — to reconstruct the tree from Example 5.1.1. This grafting encodes the equivariance of the composition structure maps, in that it restricts to the following map of fixed points:

$$\mathcal{P}(4)^{\Gamma_{G/2G\amalg G/2G}} \times \left(\mathcal{P}(3)^{\Gamma_{2G\amalg 2G/2G}} \times \mathcal{P}(3)^{\Gamma_{2G\amalg 2G/2G}}\right)^{\Gamma_{G/2G}} \longrightarrow \mathcal{P}(6)^{\Gamma_{G\amalg G/2G\amalg G/2G}},$$

as recorded earlier in 4.2.26. Moreover, we note that this grafting is suggested by the orbital representation. Indeed, grafting is defined whenever we can identify an orbital root with an orbital leaf. We will return too this discussion in Section 5.1.5.

We now highlight a particular type of G-tree, namely the corollas:

Example 5.1.8. Given any H-set $A \simeq \amalg H/K_i$, define a *corolla* for A, denoted C_A, to be any tree with an orbital representation

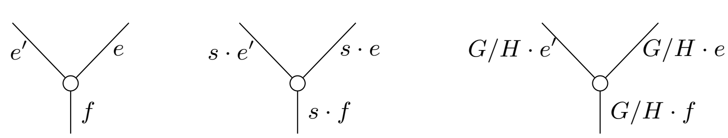


In the expanded representation, this will be a G-tree with $[G : H]$-many tree components, such that there exists a root r and a generating relation $r^{\uparrow} \leq r$ such that $\mathrm{Stab}_G(r) = H$, and $r^{\uparrow} \simeq A$ as H-sets (the other roots and leaves will give conjugations of this H-set). We will also refer to this G-tree by the tuple $(G/K_1, \ldots, G/K_n; G/H)$.

Operadically, these encode operations $\varphi \in \mathcal{P}(\Sigma[H : K_i])^{\Gamma^H_{\amalg H/K_i}} = \mathcal{P}(A)^H$.

We give some more explicit examples:

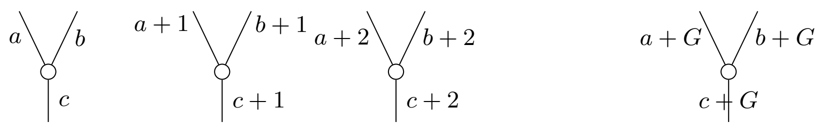
Example 5.1.9. Again, let $G = D_8$ and $H = \langle r, s^2\rangle$. Then we have the following equivariant tree, representing the trivial H-set $H/H \amalg H/H$:



This encodes operations $\varphi \in \mathcal{P}(2)^{\Gamma^H_{H/H \amalg H/H}} = \mathcal{P}(2)^H$.

Lastly, we consider “free G-corollas”.

Example 5.1.10. Let $G = Z/3$. The following equivariant tree encodes the free G-set $G \amalg G$; operadically, this encodes $\varphi \in \mathcal{P}(6)$ (more accurately, an entire orbit $G.\varphi$, but with no restrictions or relations enforced between them):



5.1.2 Forests and the Definition of Equivariant Trees

We again remark that our examples of equivariant trees above don’t look so much like trees, but instead look like *forests*, the objects in the coproduct completion of the category Ω of trees. Indeed, in this section, we will formally define G-trees as a particular class of forests.

Definition 5.1.11. A *forestly ordered set* is a finite simple broad poset F such that

(1) each edge is either a leaf, a node, or a stump; and

(2) for each edge e there is a unique $\leq_d$-maximal element $r \in T$ such that $e \leq_d r$.

We denote by $\bar{r}_F$ the set of $\leq_d$-maximal elements of F, and refer to it as the set of *roots*.

We will refer to such objects simply as “forests”, and let Φ denote the category of forests and morphisms of broad posets.

There are many possible types of morphisms we can consider between forests; however, we will restrict our discussion to the kinds of arrows which arise equivariantly. For a more complete analysis, see [Per17].

Definition 5.1.12. Given maps $f_i : T_i \to \bar{T}_i$ of trees, the coproduct $f : \amalg T_i \to \amalg \bar{T}_i$ is an arrow Φ.

Lemma 5.1.13. *A map $f : F \to \bar{F}$ of forests is a coproduct of maps of trees if whenever $f(r_{F,i}) \leq_d r_{\bar{F}}$ and $f(r_{F,j}) \leq_d r_{\bar{F}}$, we must have $i = j$. Equivalently, the map $\bar{r}_F \to \bar{r}_{F'}$ sending r to the root of the component containing $f(r)$ is bijective.* □

Definition 5.1.14. Suppose $f : T \to T'$ is an isomorphism of trees. Then, for any forest $F \in \Phi$ we have maps:

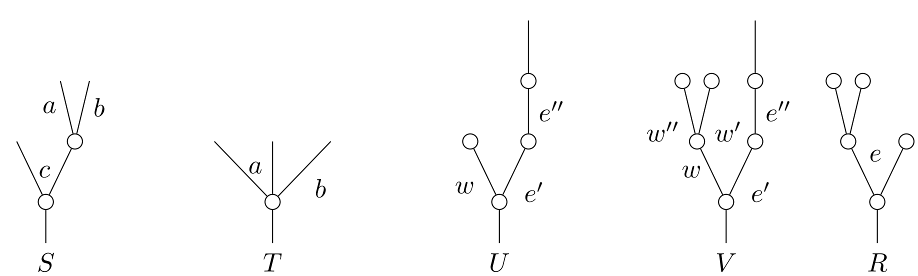
(1) $\nabla : T \amalg T' \amalg F \to T' \amalg F$ which are the identity on the F and T' components, and send T isomorphically via f to T'. Similarly, maps which combine more than two components are also in Φ. All of these are called *fold maps*.

(2) $\sigma_f : T \amalg T' \amalg F \to T \amalg T' \amalg F$ which is the identity on F, sends T to T' via f, and T' to T via f^{-1}. Similarly, maps which (twist and) permute multiples trees are in Φ, and are called *shuffles*.

Similarly to Ω, we may break down maps into faces and degeneracies.

Definition 5.1.15. A map of forests is a *degeneracy* (respectively, *face*) if it is a coproduct of degeneracy maps (respectively, the composition of coproducts of face maps and fold maps).

Example 5.1.16. Consider the following trees:



Then the following is the composition of a face map, a fold map, and a degeneracy map:

$$S \amalg T \amalg U \xrightarrow{id \amalg \partial_c \amalg \partial_{w',w''}} S \amalg S \amalg V \xrightarrow{\nabla \amalg id} S \amalg V \xrightarrow{id \amalg \sigma_{e',e''}} S \amalg R$$

The above morphisms are sufficient to define and discuss our category of equivariant trees. First, let Φ^G denote the category of forests with G-action.

Definition 5.1.17. The category of *G-trees*, denoted Ω_G, is the full subcategory of Φ^G such that the root set is a transitive G-set.

Example 5.1.18. The examples of G-trees from Section 5.1.1 clearly are G-trees under this definition.

Remark 5.1.19. At first glance, it may be odd to call such objects "trees", as they generically have an underlying forest. However, just as trees are forests which are indecomposable under the coproduct, so too equivariant trees are G-forests which are indecomposable under the coproduct. Moreover, the orbital representation (as seen in the previous section) encodes much of the data, and is itself always a tree.

It is immediate from transitivity of the root G-set that the non-equivariant tree components of a G-tree T are pairwise isomorphic in Ω.

Definition 5.1.20. Given $T_0 \in \Omega$, we say $T \in \Omega_G$ (respectively $T_H \in \Omega^H$) has *underlying shape* T_0 if all tree components of T (respectively T_H) are non-equivariantly isomorphic to T_0.

Objects in Ω_G have multiple descriptions coming from different decompositions:

Lemma 5.1.21. *Let T be a G-tree with shape T_0.*

(1) T is equivalent to a tree of the form $G \cdot_H T_H$ for some $H \leq G$ and $T_H \in \Omega^H$ of shape T_0, unique up to conjugations of H and pushforwards $g^* T_H$ of T_H by elements $g \in G$;

(2) T is isomorphic to $G \cdot T_0/N$, where N is a graph subgroup of $G \times \mathrm{Aut}(T_0)$, unique up to conjugation of N by elements of G.

Proof. These are immediate from the definition of G-tree, by choosing a component T_H of T, with H defined to be the stabilizer of the root $r_H \in T_H$, and N the graph subgroup corresponding to the homomorphism $H \to \mathrm{Aut}(T_0)$ identifying the H-action on T_H. $\square$

Remark 5.1.22. Given $T \in \Omega_G$ of shape T_0, we observe that T has a decomposition of the form $G \cdot_H T_H$ if and only if T has a decomposition of the form $G \cdot T_0/N$ with $\pi_1(N) = H$ if and only if T has an orbital representation with the root edge labeled by G/H.

Remark 5.1.23. Notationally and conceptually, we record the difference between Ω_G, the category of G-trees, and Ω^G, the category of trees with G-action. There is a natural inclusion $\Omega^G \hookrightarrow \Omega_G$ of the G-trees with a single underlying tree component; equivalently, those with a decomposition $T \simeq G \cdot T_0/N$ where N is the graph of a homomorphism $G \to \mathrm{Aut}(T_0)$.

Definition 5.1.24. We record several important functors landing in Ω_G.

(1) We have a fully-faithful inclusion $i : \Omega \times G \hookrightarrow \Omega_G$ sending $(*, T)$ to the free G-tree $G \cdot T$; that is, $\Omega \times G$ consists of the free G-trees, with all edges labeled in the orbital representation labeled by simply G. With that in mind, we will refer to objects in $\Omega \times G$ by either $(T_0, *)$ or their image $G \cdot T_0$.

(2) More generally, we have a fully-faithful functor $\Omega \times O_G \to \Omega_G$, sending $(T_0, G/H)$ to $G \cdot_H T_0$. For this reason, we often think of Ω_G as a very general amalgamation of Ω and O_G.

(3) For any $H \leq G$, we also have a fully-faithful "induction" map $\Omega^H \to \Omega_G$, sending T_H to $G \cdot_H T_H$;

We see something stronger is true.

Lemma 5.1.25. *Ω_G is equivalent to the Grothendieck construction on the functor*

$$O_G^{op} \to \mathsf{Cat}$$

which sends G/H to the category $\Omega^{B_{G/H}G}$. □

Proof. An object in the above Grothendieck construction is a G-tree with a chosen decomposition $T \simeq G \cdot_H T_H$; as isomorphisms need not preserve this choice, the result is clear. □

Denoting $G \cdot_H \eta$ by just G/H, we can repackage this result as follows.

Lemma 5.1.26. *Ω_G is equivalent to the category whose objects are G-maps of broad posets $p : T \to G \cdot_H \eta$, such that $p^{-1}(eH)$ is dendroidally ordered, and whose morphisms are generated by*

(1) maps of G-broad posets over G/H; and

(2) pullbacks over maps $q : G/K \to G/H$

where two maps are identified if they have the same underlying map of G-broad posets. □

We use the above two descriptions of Ω_G to characterize the morphisms. We see that morphisms in these latter categories can be factored uniquely (up to isomorphism) in the form

$$G \cdot_H T_H \xrightarrow{G \cdot_H f} G \cdot_H \bar{T}_H \xrightarrow{q} G \cdot_K T_K$$

with $f : T_H \to \bar{T}_H$ an arrow in $\Omega^{B_{G/H}G} \simeq \Omega^H$, and q is a Cartesian map. We discuss the two factors separately.

Faces and Degeneracies

It is clear that the face-degeneracy factorization on Ω ascends to one on Ω^H. In Ω_G, these induce face or degeneracy maps which occur in tandem over an *entire orbit* (and hence across all tree components).

Definition 5.1.27. Let $T \simeq G \cdot_H T_H$ be a G-tree.

(1) Given an outer cluster C of T_0, there is an *elementary outer G-face map*

$$G \cdot_H (\partial_{H.C} : T_H \setminus H.C \hookrightarrow T_H), \qquad \text{or equivalently} \qquad \partial_{G.C} : T \setminus G.C \hookrightarrow T;$$

(2) Given an inner edge e of T_H, we have an *elementary inner G-face map*

$$G \cdot_H (\partial_{H.e} : T_H \setminus H.e \hookrightarrow T_H), \qquad \text{or equivalently} \qquad \partial_{G.e} : T/G.e \hookrightarrow T;$$

(3) Given a unary vertex $v = (e' = e^{\uparrow} \leq e)$ of T_H, we have an *elementary G-degeneracy map*

$$G \cdot_H (\sigma_{H.v} : T_H \to T_H \setminus H.v = T_H \setminus H.\{e, e'\}),$$

or equivalently

$$\sigma_{G.v} : T \to T \setminus G.v = T \setminus G.e, e'.$$

As in the non-equivariant case, we call compositions of elementary outer G-faces (respectively, inner G-faces, G-degeneracies) *simple*, and compositions of elementary arrows and isomorphism by *outer G-faces* (*inner G-faces*, *G-degeneracies*).

The above describes what happens in the expanded representation. However, it is enlightening to consider how these maps affect the *orbital* representation. In particular, we will observe that an equivariant faces and degeneracies act on the orbital representation as *actual faces and degeneracies* on the underlying unlabeled tree (see Example 5.1.38 below).

Remark 5.1.28. One word of caution: the degeneracies (respectively, root-cluster outer faces) of the tree underlying the orbital representation which are induced by degeneracy (respectively, face) maps of trees with H-action are only those such that both edges surrounding the unital node (respectively, all edges of the root vertex) are labeled by isomorphic G-orbits. For degeneracies, this ensures that the resulting nodes in the expanded representation are actually unital; for the root-cluster outer faces, this ensures we remain in the image of Ω^H.

While this observation cannot be challenged for degeneracies, root-cluster outer face maps on the orbital representation which are not of this form are *still face maps*; however, they will be the composite of an outer G-face with a *quotient* map (Section 5.1.2 below). See Example 5.1.41 for more details.

Remark 5.1.29. Dually, elementary inner face G-maps induce face maps on the orbital notation, but the possible labelings on the new edge are restricted. Thinking of vertices as maps of G-sets, if we add an edge whose parent is labeled by G/H and whose children are labeled by $\{G/L_1, \ldots, G/L_n\}$ (and without loss of generality we may assume $L_i \subseteq H$), then the new edge must be labeled by an orbit (isomorphic to one) which factorizes these maps. That is, the new labeling G/K must have $L_1, \ldots, L_n \leq K \leq H$.

As is the case non-equivariantly, we can consider posets of simple face maps.

Definition 5.1.30. The *equivariant outer face poset* of a G-tree T (respectively, *inner face*), denoted $\mathrm{Out}^G(T)$ (resp. $\mathrm{Inn}^G(T)$), has as objects all equivariant outer (inner) faces, thought of as subsets of $E(T)$, with the relation given by inclusion.

Definition 5.1.31. The *(equivariant) core poset* of a G-tree, denoted $\mathrm{Out}_c^G(T)$, is the subposet of equivariant outer faces which are G-corollas or G-sticks.

This is equivalent to the poset of the orbits of vertices and edges, with relations generated by $G.e \leq G.v$ if $g.e$ is connected to v for some $g \in G$.

We will also consider *non-equivariant* face (resp. degeneracy) maps in Ω^H, inducing non-equivariant face (resp. degeneracy) maps in Ω_G.

Definition 5.1.32. A (non-equivariant) face of T is a map $\varphi : S_0 \to T$ of forests (with $S_0 \in \Omega$) such that φ is a face map onto its image component. These are designated *inner* or *outer* accordingly.

Equivalently, this is a map which can be factored

$$G \cdot S_0 \xrightarrow{G \cdot \varphi'} G \cdot T_0 \xrightarrow{q} G \cdot T_0/N = T,$$

for $T \simeq G \cdot T_0/N$ and $\varphi' : S_0 \to T_0$ is a face map in Ω.

We recall the outer and inner face posets $\mathrm{Out}(T_0)$ and $\mathrm{Inn}(T_0)$ associated to $T_0 \in \Omega$ (e.g. 2.3.30). Generalizing to forests $F = \amalg F_i$, define $\mathrm{Out}(F) = \amalg \mathrm{Out}(F_i)$, and similarly for $\mathrm{Inn}(F)$. In particular, for a G-tree $T \simeq G \cdot_H T_H$, $\mathrm{Out}(T)$ is isomorphic to be the G-poset $G \cdot_H \mathrm{Out}(T_H)$, and similarly $\mathrm{Inn}(T) \simeq G \cdot_H \mathrm{Inn}(T_H)$

There will be certain classes of non-equivariant face maps which "look equivariant" when translated into the orbital representation. We will naturally call these "orbital face maps".

Definition 5.1.33. A (non-equivariant) face $\varphi : S_0 \to T$ is called an *orbital face map* if $\varphi(S_0)$ is an H-closed subtree of T, where $H = \mathrm{Stab}_G(\varphi(r_S))$, r_S the root of S_0.

Denote by $\Phi_{\mathrm{Orb}}(T)$ the poset of orbital face. We will see many example of these maps in Section 5.1.3 below.

Quotient Maps

Lastly, we unpack the associated Cartesian maps. We will refer to these as *quotients* for reasons which will become apparent. Unpacking the definitions, we see that every quotient map is isomorphic to a map of the form

$$q : G \cdot_K (T_H)|_K \to G \cdot_H T_H.$$

with $K \leq H \leq G$.

We first note that on the underlying forests, these are fold maps. Second, restricting to the root orbit yields a *quotient* map $G/K \to G/H$, and this will be reflected on the orbital representation. In fact, a direct consequence of the definition of a Grothendieck fibration and Lemma 5.1.25 is the following:

Lemma 5.1.34. *Given a G-tree T with root orbit isomorphic to G/H and any G-map $f : G/K \to G/H$, there is a unique (up to isomorphism) tree S with root orbit G/K, and a quotient map $f : S \to T$.* □

See Example 5.1.38 below.

Notation 5.1.35. Let $\Omega_G^q = \Omega_{G,0}$ denote the wide subcategory of G-trees and quotient maps (including isomorphisms).

Combining Lemma 5.1.25 and Theorem 2.3.37, we have the decomposition result:

Corollary 5.1.36. *Any map $f : S \to T$ in Ω_G has a decomposition, unique up of isomorphism, as*

$$f : S \xrightarrow{\sigma} S' \xrightarrow{\varphi_i} T' \xrightarrow{\varphi_o} T'' \xrightarrow{q} T$$

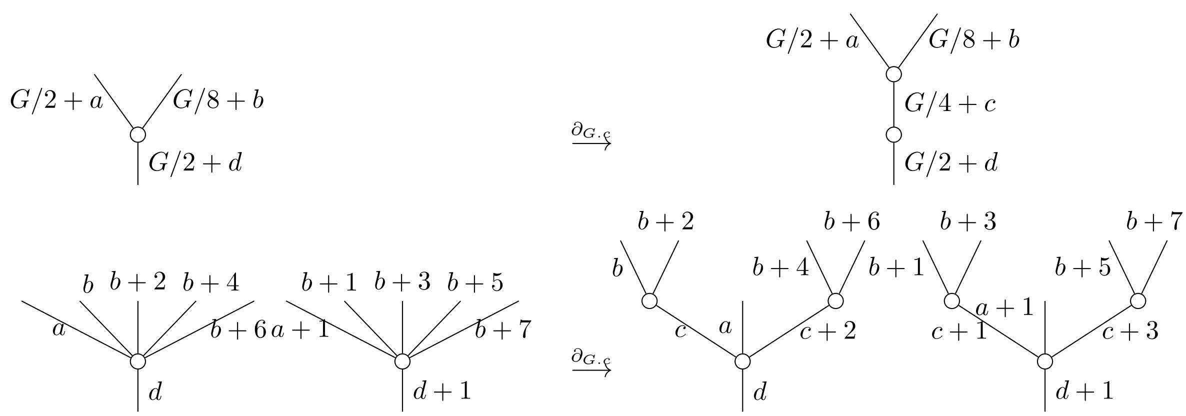
where σ is a G-degeneracy, φ_i is an inner G-face, φ_0 is an outer G-face, and q is a quotient.

□

5.1.3 Examples of Maps of Equivariant Trees

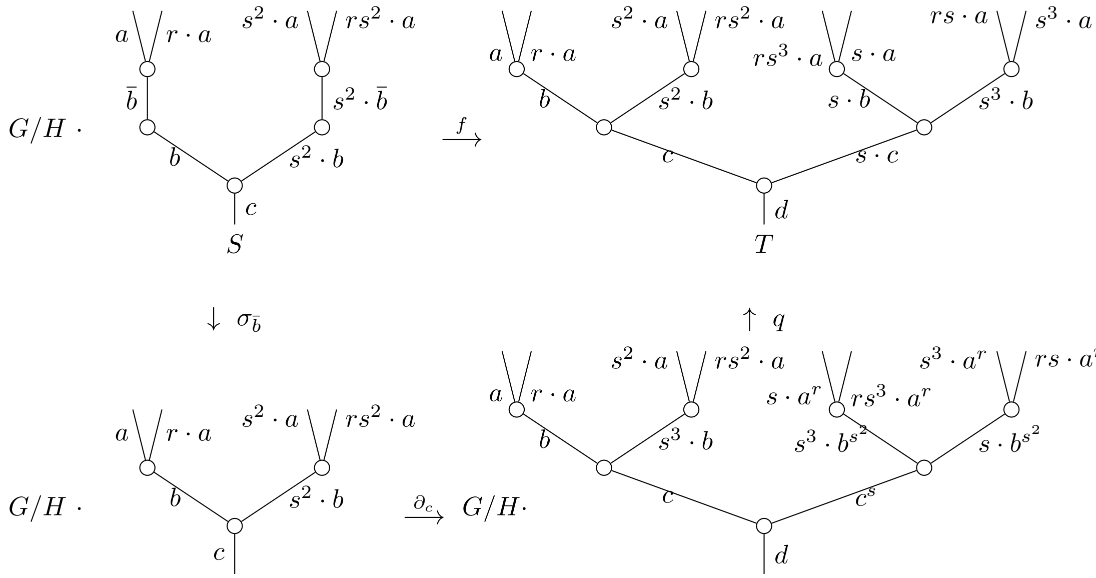
We demonstrate several maps of G-trees.

Example 5.1.37. Let $G = \mathbb{Z}/8$. The following in the inner face map $\partial_{G.c}$.

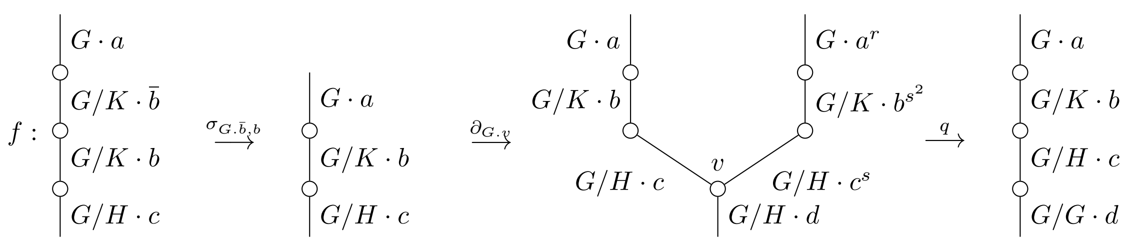


Example 5.1.38. Let $G = D_8$, $H = \langle r, s^2 \rangle$, and $K = \langle r \rangle$. Consider the following composite

$f : S \to T$, shown first in expanded representation:



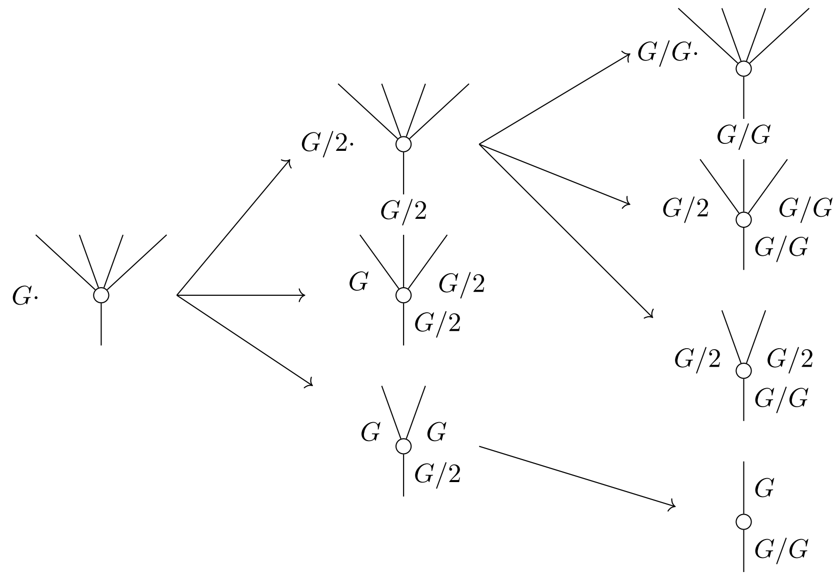
and in orbital representation:



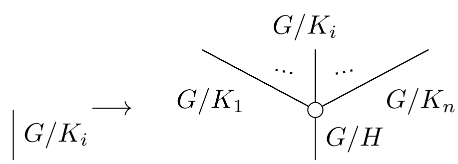
where we have labeled the edges to identify how the quotient map acts: an edge labeled e^g is in a separate orbit from e, but under q is sent to $g.e$.

Example 5.1.39. The same tree can be the source for multiple quotient maps. For example, let $G = \mathbb{Z}/4$; then the below is the subcategory (sans isomorphisms) of G-corollas with

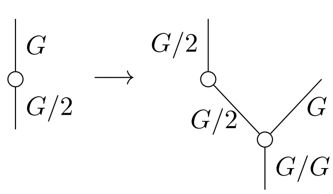
$|L(G)/G| = 4$.



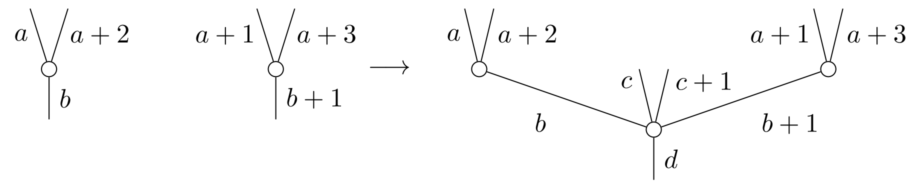
Example 5.1.40. Let A be an H-set, and C_A some A-corolla. As in the non-equivariant case, we have maps which include a stick into the edges of the corolla. However, equivariantly, this is only an outer face map if the label on the stick matches (i.e. is isomorphic to) the label on the corolla; otherwise, it is the composite of an outer face and a quotient.



Remark 5.1.41. Recalling the warning at the end of Remark 5.1.28, one may ask why we don't consider the inclusion of G-trees in orbital representation below, with $G = \mathbb{Z}/4$, an outer face map.



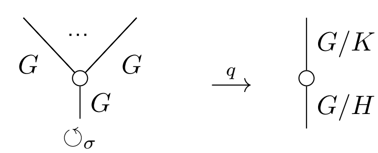
In expanded notation:



We observe that this is *not* a coproduct of maps of trees, nor is it an underlying outer face. We see that $\{a, a+2 \leq b\}$ into T_0 is a (non-equivariant) outer face, but this cannot be extended to a G-fixed outer face on T_0. Further, we can factor this map as an outer face followed by a quotient (similarly to those in Example 5.1.38).

As this does look like a face map on orbital representations, we will call this a "face" map, just not an outer or inner face.

Remark 5.1.42. We recall that the original Ω is homotopically a particularly nice diagram category, in that it's *dualizably Reed*: both Ω and Ω^{op} are *generalized Reedy categories* [BM11]. While Ω_G is Reedy (with Ω_G^+ the faces and quotients, and Ω_G^- the degeneracies), it is *not* dualizably so unless $G = \{e\}$ is the trivial group. For example, given any non-trivial H/K, consider the quotient



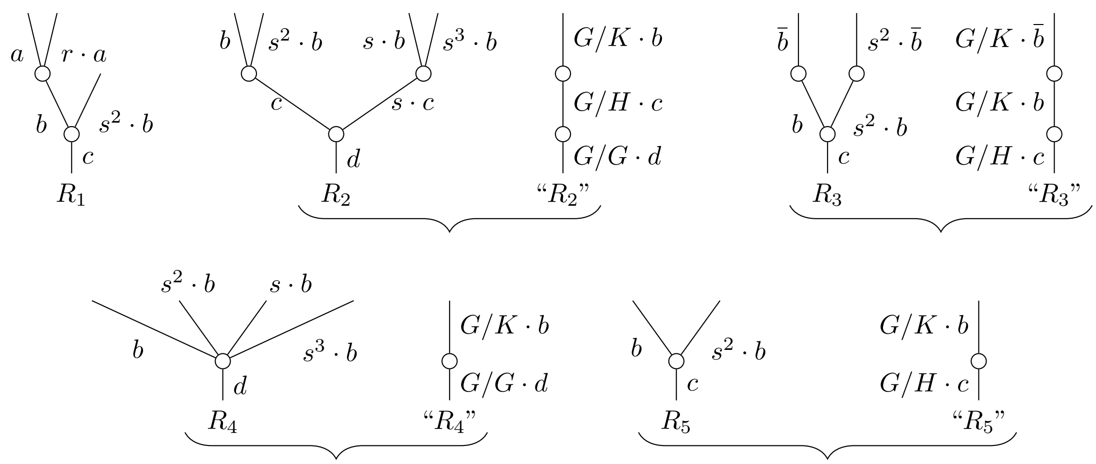
where the first tree has $n = [H : K]$ many leaves. We note that Σ_n acts on this first tree by permuting the leafs. Moreover, $q\sigma = q$ for any $\sigma \in \Sigma_n$, violating [BM11, Definition $(iv)'$] with $f = q$ and $\theta = \sigma$ non-trivial.

In the generality of $\mathcal{F}$-trees defined in the following section, $\Omega_{\mathcal{F}}$ will not be dualizably Reedy as soon as any non-trivial H-set is admissible for any $H \leq G$.

Non-Equivariant Faces

We recall that non-equivariant faces are just underlying face maps $\varphi : S_0 \to T$ (equivalently, $G \cdot S_0 \to T$ with specified unit component of the source). As such, their actual orbital representation is often fairly unenlightening.

Example 5.1.43. Recall the D_8-trees S and T from Example 5.1.38. We consider several non-equivariant maps into T below, where edges have been labeled by their image in T; we again remind ourselves that these source maps have the *free* G-action, and what it written below is just the image of the component corresponding to the unit $e \in G$.



where we have

- R_1 is non-orbital into T;
- R_2 is an orbital leaf-cluster outer face of T;
- R_3 is an orbital degeneracy of S
- R_4 is an orbital inner face of T; and
- R_5 is an orbital root-cluster outer face of T.

In particular, the "R_i" orbital representations are not in fact recording the orbital representation of the trees (as they are all free), but instead are recording the associated "face or degeneracy of the orbital representation".

5.1.4 G-Corollas

Let us consider the category of G-corollas.

Definition 5.1.44. Let Υ_G denote the full subcategory of $\Omega_G^q = \Omega_{G,0}$ spanned by all equivariant corollas.

As is true non-equivariantly, Υ_G is *almost* a full subcategory of Ω_G; however, we are excluding degeneracies, in particular maps out of $C_{H/H}$ which factor through the degeneracy $C_{H/H} \to H/H \cdot \eta$.

Lemma 5.1.45 (c.f Lemma 5.1.25, Lemma 5.1.26)**.** *The following categories are equivalent:*

(1) the category of G-corollas Υ_G;

(2) the Grothendieck construction on the functor

$$\begin{aligned} O_G^{op} &\longrightarrow \mathsf{Cat} \\ G/H &\longmapsto \Upsilon^{B_{G/H}G} \end{aligned}$$

(3) the subcategory of arrows in Set^G spanned by those of the form $A \to G/H$, for any H, with only pullback squares as morphisms.

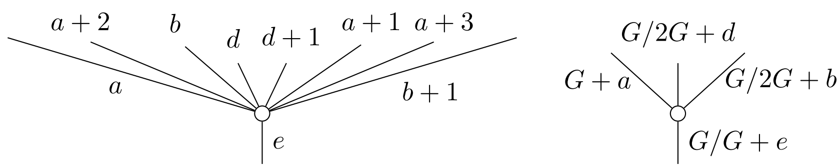
$\square$

Remark 5.1.46 (c.f. Example 5.1.8)**.** We have a surjection $\coprod_{H \leq G} \mathrm{Ob}(\mathsf{Set}^H / \simeq) \to \mathrm{Ob}(\Upsilon_G / \simeq)$, where in either case $A \simeq B$ if A is isomorphic to B; two sets are set to the same G-corolla if and only if they are conjugate.

As is the case non-equivariantly, we have a (non-planar) "valence" functor $\Omega_G \to \Upsilon_G$. Abstractly, we have valence functors $val^H : \Omega^H \to \Upsilon^H$, and hence the equivariant val exists by Lemmas 1.1.11, 5.1.25 and 5.1.45.

Explicitly, this maps sends the G-tree T to the G-broad poset with objects $\bar{r}_T \amalg L(T)$, and generating (actually, only non-trivial) relations $r^\lambda \leq r$ for all $r \in \bar{r}_T$. It is immediate that this G-broad poset is in fact a G-corolla. With this description, it may also be referred to as the *leaf-root functor*.

Example 5.1.47. Recall the tree from Example 5.1.1. It's image under val is the following G-corolla:



The inclusion map $s : \Upsilon_G \to \Omega_G$ is a section of val. We note that this inclusion can also be built the synthesizing the various inclusions $s : \Upsilon^H \hookrightarrow \Omega^H$.

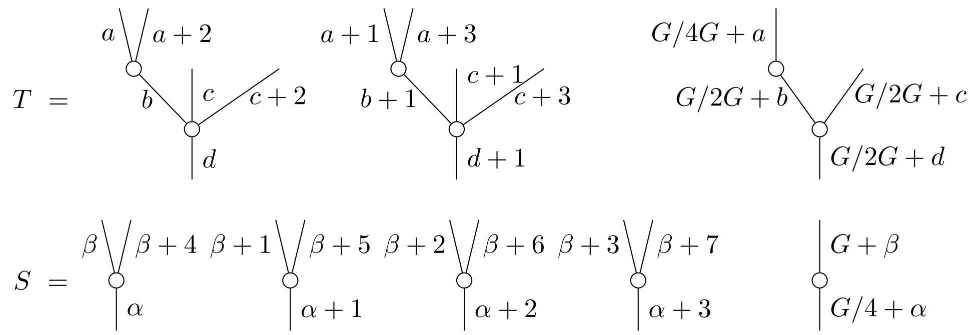
5.1.5 Grafting and $\mathcal{F}$-Trees

We update the notion of grafting to the category of G-trees.

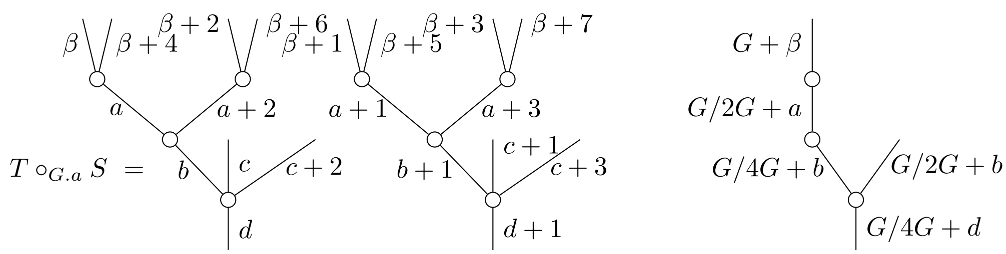
Definition 5.1.48. Let T be a G-tree, and $G.e \simeq G/H$ and an orbit of leaves. Given another G-tree S with an isomorphism $\bar{r}_S \simeq G.e$, define the grafting $T \circ_{G.e} S$ to be the G-broad poset with edges $T \amalg_{G.e} S$, and with generating broad relations as in T and S; it is clear that this resulting object is in fact a G-tree.

This operation can be visualized as a grafting on the orbital representation of our G-trees along leaf edge of T and the root edge of S which have isomorphic labels.

Example 5.1.49. Let $G = \mathbb{Z}/8$, and consider the G-trees T and S below:



We may graft S onto T via the isomorphism $\alpha \mapsto a$, yielding the following G-tree:



As in the non-equivariant case, we have a G-tree decomposition via grafting:

Lemma 5.1.50. *If T is a G-tree with root corolla C_A such that $|L(C)/G| = n$ (equivalently, A is the disjoint union of n different orbits), then $T \simeq C_A \circ (T_1, \ldots, T_n)$ for some G-trees T_i.* $\square$

Now, we saw above in Section 4.2.2 the notion of an indexing system. These provide examples of useful subcategories of Ω_G, by restricting the types of nodes we see in the orbital representation. In fact, more general notion, inspiring by the grafting of G-trees, will work in this context.

Definition 5.1.51. Let $\mathcal{F}$ be a sub coefficient system of $\underline{\mathsf{Set}}$. We recall that an H-set A is called *$\mathcal{F}$-admissible* if $A \in \mathcal{F}(G/H)$.

(1) A G-corolla C is called *$\mathcal{F}$-admissible* if, for any (equivalently, all) roots e of C, the

$\mathrm{Stab}_G(e)$-set $e^{\uparrow}$ is $\mathcal{F}$-admissible. Equivalently, for any choice of orbital representation $(G/K_1, \ldots, G/K_n; G/H)$ of C, we have that $\amalg H/K_i$ is $\mathcal{F}$-admissible.

(2) A G-tree $T \in \Omega_G$ is called *$\mathcal{F}$-admissible* if for each vertex (i.e. generating relation) $e^{\uparrow} \leq e$, the $\mathrm{Stab}_G(e)$-set $e^{\uparrow}$ is $\mathcal{F}$-admissible; equivalently, each node in any orbital representation is an $\mathcal{F}$-admissible corolla.

(3) Given $T_0 \in \Omega$, we call a graph subgroup $N \leq G \times \mathrm{Aut}(T_0)$ *$\mathcal{F}$-admissible* if the induced G-tree $G \cdot T_0/N$ is $\mathcal{F}$-admissible. We denote by $\mathcal{F}_{T_0}$ the family of all $\mathcal{F}$-admissible graph subgroups.

Remark 5.1.52. We will return to the characterization of the family $\mathcal{F}_{T_0}$ of graph subgroups of $G \times \mathrm{Aut}(T_0)$ in Section 5.3.

Definition 5.1.53. Let $\mathcal{F}$ be a sub coefficient system of $\underline{\mathsf{Set}}$.

(1) We say $\mathcal{F}$ is closed under *broad self-induction* if whenever $\amalg_i H/K_i \amalg H/K_0$ and $\amalg_j K_0/L_j$ are $\mathcal{F}$-admissible, then so is $\amalg_i H/K_i \amalg \amalg_j H/L_j$.

(2) We say $\mathcal{F}$ is closed under *grafting induction* if for all $\mathcal{F}$-admissible trees T, the G-corolla $val(T)$ is also $\mathcal{F}$-admissible.

Lemma 5.1.54. *$\mathcal{F}$ is closed under broad self-induction if and only if it is closed under grafting induction.*

Proof. It is clear that grafting induction along G-trees with $|V_G(T)| = 2$ (i.e. a single internal edge orbit) is equivalent to broad induction. Moreover, by induction along the grafting decomposition from Lemma 5.1.50, this is sufficient to generate all grafting induction. $\square$

This inspires a (weaker) notion of a "well-behaved" sub-system of $\underline{\mathsf{Set}}$.

Definition 5.1.55. A *weak indexing system* (cf. 4.2.17) is a sub coefficient system $\underline{\mathcal{F}}$ of $\underline{\mathsf{Set}}$ that contains all trivial orbits H/H for $H \leq G$, and is closed under grafting induction.

Lemma 5.1.56. *Weak indexing systems are closed under conjugation, restriction, and products with an orbit.*

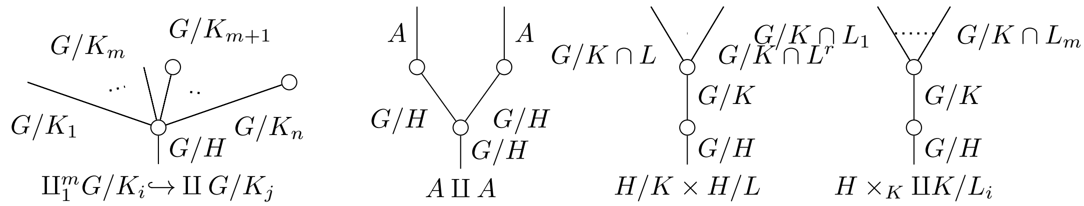
Proof. Closure under conjugation and restriction is necessary for $\mathcal{F}$ to be a sub-coefficient system of $\underline{\mathsf{Set}}$. Now, if $\amalg H/K_i$ and H/L are admissible, consider the corolla which encodes $H/L|_{K_i}$ for each i; as $\mathcal{F}$ is closed under restriction, these are admissible. Now, if we graft these on top of the corolla encoding $\amalg H/K_i$ (e.g. third tree in Diagram (5.3)), this will yield a tree whose image under *val* is precisely the corolla encoding $(\amalg H/K_i) \times H/L$, as specified by the double coset formula. □

We note that weak indexing systems are not necessarily closed under coproducts or subobjects, in particular, as $\mathcal{F}$ need not contain the *empty* H-set, nor any trivial H-set with more than one element, for any $H \leq G$.

Lemma 5.1.57. *A weak indexing system is actually a (strong) indexing system if it contains all trivial H-sets for all $H \leq G$.*

Proof. As self-induction is a particular kind of grafting induction (e.g. fourth tree in 5.3), it suffices to show that such weak indexing systems are closed under subobjects, products, and coproducts. Subobjects can be created by selectively grafting equivariant 0-corollas on particular leaf orbits and applying grafting induction (first tree in 5.3), while the coproduct of the H-sets $A_1, \ldots, A_n$ can be created by grafting each C_{A_i} onto the corolla $C_{H/H^{\amalg n}}$ encoding the trivial H-set of cardinality n (second tree in 5.3). Lastly, since weak indexing systems are closed under products with an orbit, and general products are coproducts of these, we are done. □

The grafting operations used in the proofs of the above two lemmas are displayed in the orbital representation below:



(5.3)

Definition 5.1.58. Let $\mathcal{F}$ be a weak indexing system. Define $\Omega_{\mathcal{F}}$ to be the full subcategory of Ω_G spanned by $\mathcal{F}$-admissible trees, and $\Upsilon_{\mathcal{F}}$ the full subcategory of $\mathcal{F}$-admissible corollas. Here, the set of objects of $\Upsilon_{\mathcal{F}}$ is in bijection with the set of $\mathcal{F}$-admissible sets, modulo isomorphisms.

Precisely since $\mathcal{F}$ is closed under grafting induction, the valence functor restricts to a functor $val : \Omega_{\mathcal{F}} \to \Upsilon_{\mathcal{F}}$.

Lemma 5.1.59. *Suppose we have a map $f : S \to T$ of G-trees, and $\mathcal{F}$ a weak indexing system.*

(1) If T is $\mathcal{F}$-admissible, and f is an inner or outer face map, then S is also $\mathcal{F}$-admissible;

(2) If T is $\mathcal{F}$-admissible, and f is a quotient, then S is also $\mathcal{F}$-admissible;

(3) If f is a degeneracy, then S is $\mathcal{F}$-admissible if and only if T is.

Proof. Parts (i) and (iii) are clear, since inner faces and degeneracies do not affect the image under val, and $\mathcal{F}$-admissibility is determined on vertices. For (ii), we observe that if $e^{\uparrow} \leq e$ is a vertex in T and $\bar{e} \in f^{-1}(e)$, we have that $\bar{e}^{\uparrow} = e^{\uparrow}|_{\mathrm{Stab}_G(\bar{e})}$. Hence, since $\mathcal{F}$ is closed under restriction, all vertices are $\mathcal{F}$-admissible, and thus so is S. $\square$

In fact, these properties *characterize* weak indexing systems $\mathcal{F}$:

Lemma 5.1.60. *$\mathcal{F}$ is a weak indexing system if and only $\Omega_{\mathcal{F}}$ is a sieve of Ω; that is, given any $f : S \to T$ in Ω_G, if $T \in \Omega_{\mathcal{F}}$ then both S and the map are also in $\Omega_{\mathcal{F}}$.*

Example 5.1.61. These $\Omega_{\mathcal{F}}$ capture most of the relevant subcategories of Ω_G:

(1) If $\mathcal{F} = \underline{\mathsf{Set}}$, then $\Omega_{\mathcal{F}} = \Omega_G$.

(2) If $\mathcal{F} = \mathcal{F}_{\varnothing}$ is the indexing system with only the trivial $\{e\}$-sets, then $\Omega_{\mathcal{F}} = \Omega \times G$.

(3) If $\mathcal{F} = \mathcal{F}_{\text{triv}}$ is the indexing system of all trivial sets, then $\Omega_{\mathcal{F}} \simeq \Omega \times O_G^{op}$.

(4) If $\mathcal{F} = \mathcal{F}_{\Delta}$ is the weak indexing system with all trivial sets of cardinality one, then $\Omega_{\mathcal{F}} \simeq \Delta \times O_G^{op}$.

5.2 $\mathcal{F}$-Equivariant Dendroidal Sets

We now define our equivariant generalization of dendroidal sets, using the above notions of G-trees. In particular, we have models of equivariant dendroidal sets for each weak indexing system $\mathcal{F}$.

5.2.1 The Presheaf Categories

We now consider various presheaf categories indexed by the $\Omega_{\mathcal{F}}$ of Section 5.1.5 and 5.1.5.

Definition 5.2.1. Define the category of *$\mathcal{F}$-equivariant dendroidal sets*, denoted $\mathsf{dSet}_{\mathcal{F}}$, to be the presheaf category $\mathsf{Set}^{\Omega_{\mathcal{F}}^{op}}$. Given any G-tree T, let $\Omega_{\mathcal{F}}[T]$ denote the representable presheaf

$$\Omega_{\mathcal{F}}[T](S) = \Omega_G(S, T).$$

As $\Omega_{\mathcal{F}}$ will always be a restriction of Ω_G, we will abuse notation and just write $\Omega[T]$ whenever the underlying $\mathcal{F}$ is unambiguous.

The main two examples we will consider are the extremes:

- *dendroidal sets with G-action* $\mathsf{dSet}^G := \mathsf{dSet}_{\mathcal{F}_{\varnothing}}$; and
- *genuine equivariant dendroidal sets* $\mathsf{dSet}_G := \mathsf{dSet}_{\underline{\mathsf{Set}}}$.

Given an inclusion $j : \mathcal{F} \hookrightarrow \bar{\mathcal{F}}$ of weak indexing systems, we have an associated diagram

$$\begin{tikzcd} \mathsf{dSet}_{\mathcal{F}} \arrow[r, bend left, "j_!"] \arrow[r, bend right, "j_*"'] & \mathsf{dSet}_{\bar{\mathcal{F}}} \arrow[l, "j^*"'] \end{tikzcd}$$

with $(j_!, j^*)$ and (j^*, j_*) pairs of adjoint functors.

In the case $j = i : \mathcal{F}_\varnothing \hookrightarrow \underline{\mathsf{Set}}$, we can be more explicit: given $X \in \mathsf{dSet}^G$ and $Y \in \mathsf{dSet}_G$, we have

$$i_! X(G \times T_0/N) \simeq X(T_0, *) \text{ if } N = \{e\}, \text{ and } \varnothing \text{ otherwise} \tag{5.4}$$

$$i_* X(G \times T_0/N) = \mathrm{Hom}(i^* \Omega_G[G \cdot T_0/N], Y) \simeq Y(T_0, *)^N \tag{5.5}$$

$$i^* Y(T_0, *) = Y(G \cdot T_0) \tag{5.6}$$

It is easily directly verified that the formula for $i_* X$ yields a right adjoint to i^*. In particular, we use that any map $Y(G \cdot S_0/M) \to X(S_0, *)^M$ factors through $Y(G \cdot S_0)^M$:

$$\begin{tikzcd} \mathsf{dSet}_\mathsf{G}(\Omega_G[G \cdot S_0/M], Y) \arrow[r] \arrow[d] & \mathsf{dSet}_G(\Omega_G[G \cdot S_0/M], i_* X) \arrow[d, equal] \\ \mathsf{dSet}_\mathsf{G}(\Omega_G[G \cdot S_0]/M, Y) \arrow[r] & \mathsf{dSet}_G(\Omega_G[G \cdot S_0]/M, i_* X). \end{tikzcd}$$

More, as each $j : \Omega_{\mathcal{F}} \to \Omega_{\bar{\mathcal{F}}}$ is fully-faithful, we have the following.

Lemma 5.2.2. *The functors j_* and $j_!$ are both sections of j^*; hence $\mathsf{dSet}_{\mathcal{F}}$ is both a reflective and coreflective subcategory of $\mathsf{dSet}_{\bar{\mathcal{F}}}$ whenever $\mathcal{F} \subseteq \bar{\mathcal{F}}$.* □

Remark 5.2.3. The condition expressed in (5.5) should be interpreted analogously to the condition that in the fixed-point coefficient system ΦX of a G-space X, we have $\Phi X(G/H) = \Phi X(G/e)^H$. We will come back to this idea in a later section, exploring the image of nerve functors.

Lemma 5.2.4. *For any G-tree T and inclusion $j : \mathcal{F} \to \bar{\mathcal{F}}$ of weak indexing systems, we have*

(1) $j_*\Omega_{\mathcal{F}}[T] \simeq \Omega_{\bar{\mathcal{F}}}[T]$, *and*

(2) $j^*\Omega_{\bar{\mathcal{F}}}[T] = \Omega_{\mathcal{F}}[T]$.

Proof. Since $\Omega_{\bar{\mathcal{F}}}[T] = (j_G)^*\Omega[T]$ for $j : \Omega_{\bar{\mathcal{F}}} \hookrightarrow \Omega_G$, it suffices to show that $(j_G)_* : \mathsf{dSet}^G \to \mathsf{dSet}_G$ sends $\Omega[T]$ to $\Omega[T]$. But we have

$$(j_G)_*\Omega[T](G \cdot S_0/N) = \Omega[T](S_0, *)^N = \Omega_G(G \cdot S_0, T)^N = \Omega_G(G \cdot S_0/N, T).$$

$\square$

Remark 5.2.5. We record certain parallels with the definitions of orthogonal G-spectra, in that there is a choice of how much equivariance is encoded in the indexing. In [HHR16], it is shown that working with the complete indexing category of a complete G-universe is often the most practically useful, while [Lew95, EM97, MM02] show that all possible choices of categories are equivalent, and in fact [Sto11, HW13] show that for any universe U there exist model structures on all possible categories which are Quillen equivalent to the natural U-model structure on U-indexed spectra.

Homotopically, we expect this situation to be similar. Again, we have categories $\mathsf{dSet}_{\bar{\mathcal{F}}}$ for any choice of weak indexing system $\bar{\mathcal{F}}$ — though these categories will not be equivalent; in particular, objects in $\mathsf{dSet}_{\mathcal{F}}$ do not have strict fixed-point conditions when evaluated on $\bar{\mathcal{F}}$-admissible sets, and this certainly induces a categorical dependence on $\bar{\mathcal{F}}$. However, as we will see in Theorem 5.2.23, $\mathsf{dSet}_{\mathcal{F}}$ has an $\mathcal{F}'$-model structure for any other indexing family $\mathcal{F}'$, which we fully expect to be Quillen equivalent to the $\mathcal{F}'$-model structure on any other $\mathsf{dSet}_{\bar{\mathcal{F}}'}$.

Below the minimum equivariant level dSet^G, we have the natural inclusion $i_G : \Omega \hookrightarrow \Omega \times G$, inducing a diagram

$$\begin{tikzcd} \mathsf{dSet} \arrow[r, bend left=40, "(i_G)_!"] \arrow[r, bend right=40, "(i_G)_*"'] & \mathsf{dSet}^G \arrow[l, "(i_G)^*"'] \end{tikzcd}$$

where again $((i_G)_!, (i_G)^*)$ and $((i_G)^*, (i_G)_*)$ are adjoint pairs. Explicitly, given $W \in \mathsf{dSet}$ and $X \in \mathsf{dSet}^G$, we have:

$$i_! W(T_0, *) = G \times W(T_0)$$
$$i_* W(T_0, *) = \mathsf{Set}(G, W(T_0))$$
$$i^* X(T_0) = Y(T_0, *)$$

Unpacking definitions, this yields the following descriptions of $\Omega[T]$ in dSet^G.

Lemma 5.2.6. *Given $T \in \Omega_G$ with decompositions $T \simeq G \cdot T_0/N \simeq G \cdot_H T_H$, we have*

$$\begin{aligned} \Omega[T] \simeq i^* \Omega_G[T] &\simeq i^* \Omega_G[G \cdot T_0]/N \\ &\simeq G \times_H \Omega[T_H] \\ &\simeq (i_G)_! \Omega[T_0]/N \end{aligned}$$

where here $\Omega[T_H]$ is just $\Omega(-, T_H)$ with the induced H-action. □

Remark 5.2.7. There is another perspective to view the the "initial" representable functor $\Omega[T]$. We recall the non-equivariant Yoneda embedding $\Omega[-] : \Omega \hookrightarrow \mathsf{dSet}$. This naturally extends to a map out of the category Φ of forests, sending $F = \amalg T_i$ to $\amalg \Omega[T_i]$. Passing to the categories of G-objects, and restricting along the inclusion $\Omega_G \hookrightarrow \Phi^G$, yields a map

$$\Omega[-] : \Omega_G \to \mathsf{dSet}^G,$$

for which is it easy to check is isomorphic to $\Omega[-] = \Omega_{\mathcal{F}_\varnothing}$ above.

For the remainder of this section, we will mainly be focusing on dSet^G. While dSet_G is the most interesting of these categories, in that it has the most flexibility and records the most data (in particular, the nerve functor on *genuine equivariant operads* of Chapter 6 will land here), it has some technical difficulties which will require some future manipulation to overcome (e.g. Remark 5.1.42).

5.2.2 Faces, Boundaries, and Horns

We will now build the standard presheaves used to define model structures and various other important constructions in a generalized Reedy categories or EZ-categories, such as boundaries and horns.

For this section, while we will be working mostly in dSet^G, we fix a (new) weak indexing system $\mathcal{F}$ which we will use to generate the components of the $\mathcal{F}$-model structure on dSet^G. The most important example will be the complete indexing system $\mathcal{F} = \underline{\mathsf{Set}}$; in the below, to refer to this particular example, we replace all instances of "$\mathcal{F}$" with "G" (e.g. G-inner horn inclusions, G-normal monomorphisms, G-∞-operads).

We will now describe several different classes of maps in dSet^G. These will play a similar role as the classes with similar names in [CM11].

Definition 5.2.8. Given $X \in \mathsf{dSet}^G$ and $T_0 \in \Omega$, we call $x \in X(T_0, *)$ *degenerate* if the characterizing map $\Omega[G \cdot T_0] \to X$ factors

$$\Omega[G \cdot T_0] \xrightarrow{\sigma} \Omega[G \cdot S_0] \to X$$

where $\sigma : T_0 \to S_0$ is a degeneracy in Ω. Otherwise, we call x *non-degenerate*.

Definition 5.2.9. A monomorphism $f : X \to Y$ in dSet^G is call $\mathcal{F}$*-normal* if for all $T_0 \in \Omega$ and non-degenerate $y \in Y(T_0, *) \backslash X(T_0, *)$, the stabilizer $\mathrm{Stab}_{G\times\mathrm{Aut}(T_0)}(y)$ is a $\mathcal{F}$-admissible graph subgroup of $G \times \mathrm{Aut}(T_0)$.

In order to define the boundaries in dSet^G, we add to our discussion about non-equivariant faces at the end of Section 5.1.2.

Definition 5.2.10. Given $T \in \Omega_G$, $R_0 \in \Omega$, we recall that a non-equivariant face is just a map $\varphi : R_0 \to T$ of forests which is a face map on the image component. We call the associated map $\varphi : \Omega[G \cdot R_0] \to \Omega[T]$ a *face* of $\Omega[T]$. We denote by $\partial_\varphi \Omega[T] \in \mathsf{dSet}^G$ the image of this map in $\Omega[T]$. Explicitly, we have

$$\partial_\varphi \Omega[T](S_0, *) = \{f : \Omega[G \cdot S_0] \to \Omega[T] \mid f \text{ factors through } \varphi\}.$$

Definition 5.2.11. The *boundary inclusion* of $\Omega[T]$ is the map

$$\partial\Omega[T] := \operatorname*{colim}_{\varphi \in \mathrm{Inn}(T)} \partial_\varphi \Omega[T].$$

Non-equivariantly, we can also define boundaries on forests $F = \amalg F_i$ by $\partial\Omega[F] = \amalg \partial\Omega[F_i]$. In fact, these definition agree:

Lemma 5.2.12. *Given any decompositions $T \simeq G \cdot T_0/N \simeq G \cdot_H T_H$, we have that the boundary inclusion for T is equivalent to both*

$$i_!(\partial\Omega[T_0] \hookrightarrow \Omega[T_0])/N$$

and

$$G \cdot_H (\partial\Omega[T_H] \hookrightarrow \Omega[T_H]),$$

where again $\Omega[T_H] \in \mathsf{dSet}^{\mathsf{H}}$. □

Proof. This is immediate after unpacking definitions. We state the first correspondence explicitly; the second is similar. We note that $\mathsf{dSet}^G(\partial\Omega[T], X)$ is equivalent to tuples (x_φ) of elements $x_\varphi \in X(G \cdot R_0)$, $\varphi : R_0 \to T$ a elementary face map (of forests), with the φ compatible over equalizing maps. Conversely, maps $\mathsf{dSet}^G((i_G)_!\partial\Omega[T_0]/N, X)$ is similarly equivalent to tuples $(x_\varphi) \in i_G^* X(R_0) = X(G \cdot R_0)$ for all elementary face maps $\varphi : R_0 \to T_0$ which are "N-equivariant" — that is, $x_\varphi = x_{\bar{\varphi}}$ if $T_0 \to T$ equalizes φ and $\bar{\varphi}$. Thus both objects define the same representable functor, and hence are isomorphic. □

We will define two types of horns, *underlying* and *orbital*, each capitalizing on a different aspect of Ω_G. We begin with the former, which are more standard.

In either case, we quickly realize that elementary inner horns will very rarely be equivariant: if $e \in T$ is an inner edge, then $\Lambda^e[T]$ is an H-object if and only if e is H-fixed. To fix these peculiarities, we will need to consider horns over an entire *orbit's* worth of edges. To that end, we first recall (non-equivariant) generalized inner horns (see [MW09, Section 8]).

Definition 5.2.13. If A is a set of inner edges of some $T_0 \in \Omega$, define $\Lambda^A[T_0]$ to be the union of all simple faces not of the form $\partial_{A'}$ for $A' \subseteq A$.

More generally, if $F = \amalg T_i$ is a forest, and $A = \amalg A_i$ for A_i a set of inner edges of T_i, define

$$\Lambda^E[F] := \amalg \Lambda^{E_i}[T_i].$$

Definition 5.2.14. The *elementary inner $\mathcal{F}$-horn inclusions* are maps in dSet^G of the form

$$\Lambda^{G.e}[T] \hookrightarrow \Omega[T]$$

where T is an $\mathcal{F}$-admissible G-tree and $G.e$ is the G-orbit of some inner edge e.

Remark 5.2.15. We note that $\Lambda^{G.e}[T]$ is in fact an object in dSet^G. Indeed, $\Lambda^{G.e}[T](S_0, *)$ is the set of maps $\Omega[G \cdot S_0] \to \Omega[T]$ that factor through a face map which misses $G.e$; precisely because $G.e$ is a full orbit, this set has the appropriate G-action.

The following result is proved analogously as to Lemma 5.2.12.

Lemma 5.2.16. *Elementary inner $\mathcal{F}$-horn inclusions are isomorphic to maps of the form*

$$i_!(\Lambda^{H.e}[T_0] \hookrightarrow \Omega[G \cdot T_0])/N$$

where N is an $\mathcal{F}$-admissible graph subgroup of $G \times \mathrm{Aut}(T_0)$ with $\pi_1(N) = H$. $\square$

Definition 5.2.17. We call a map in dSet^G *inner $\mathcal{F}$-anodyne* if it is in the saturation of the set of elementary inner $\mathcal{F}$-horn inclusions under retracts of transfinite compositions of pushout.

We now introduce our combinatorial models for "G-homotopical operads".

Definition 5.2.18. A dendroidal set $X \in \mathsf{dSet}^G$ is called an $\mathcal{F}$-∞-*operad*, or an *inner $\mathcal{F}$-Kan complex*, if X has the right lifting property with respect to all elementary inner $\mathcal{F}$-horn inclusions.

We call such an X *strict* if it has unique liftings.

More generally, we call a map a (strict) *inner $\mathcal{F}$-fibration* if it has the (strict) right lifting property with respect to all elementary inner $\mathcal{F}$-horn inclusions.

It is clear that if $\mathcal{F}' \hookrightarrow \mathcal{F}$ is an inclusion of weak indexing systems and $f : X \to Y$ is an inner $\mathcal{F}$-fibration, then f is also an inner $\mathcal{F}'$-fibration.

We can extend these definitions to the other presheaf categories besides dSet^G:

Definition 5.2.19. If $\bar{F}$ is another weak indexing system, then an arrow $f : X \to Y$ in $\mathsf{dSet}_{\bar{\mathcal{F}}}$ is called an *(strict) inner $\mathcal{F}$-fibration* if i^*f is an (strict) inner $\mathcal{F}$-fibration in dSet^G.

The following observations are straightforward:

Lemma 5.2.20. *Suppose we have inclusions $\bar{\mathcal{F}}' \xrightarrow{j} \bar{\mathcal{F}} \xrightarrow{j'} \bar{\mathcal{F}}''$ of weak indexing systems, and an arrow $f : X \to Y$ in $\mathsf{dSet}_{\bar{\mathcal{F}}}$. The following are equivalent:*

(1) f is an inner $\mathcal{F}$-fibration.

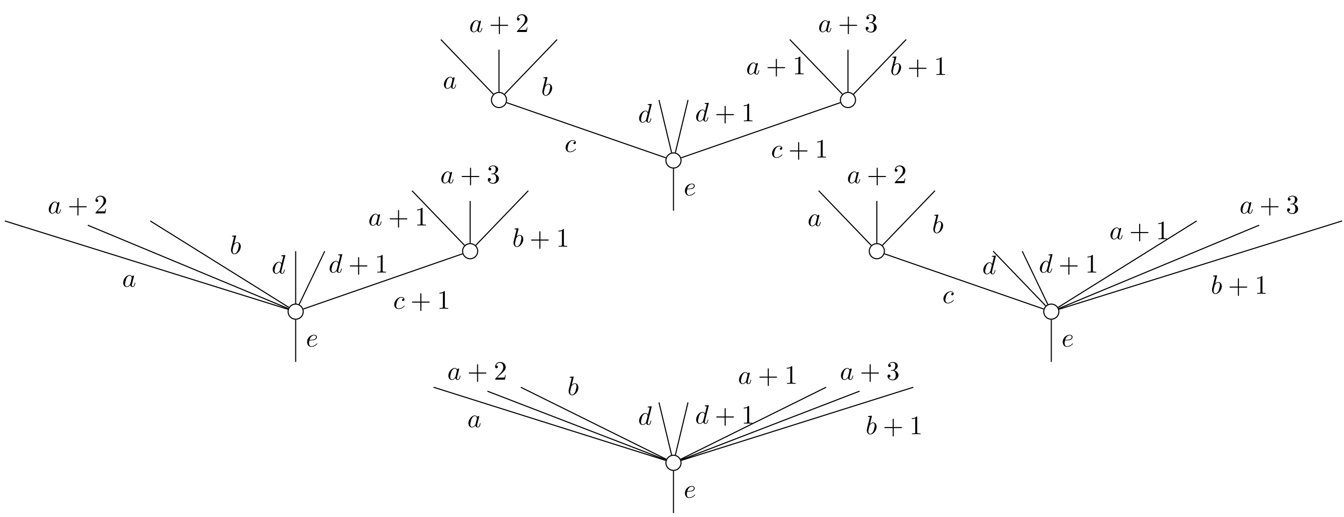
(2) $j^*f \in \mathsf{dSet}_{\bar{\mathcal{F}}'}$ is an inner $\mathcal{F}$-fibration.

(3) $j_*f \in \mathsf{dSet}_{\bar{\mathcal{F}}''}$ is an inner $\mathcal{F}$-fibration.

□

Remark 5.2.21. As the ∞-operads of Moerdijk-Weiss and Cisinski-Moerdijk have non-canonical composition of operations over trees, G-∞-operads have "non-canonical composition over G-trees". Additionally, they can be strictified to equivariant operads. More completely, they can be strictified to "genuine equivariant operads", a algebraic construction designed to capture the information inherent in a G-∞-operad. We will explore this connection further in Chapter 6.

Example 5.2.22. Let $G = \mathbb{Z}/4$, $\mathcal{F} = \underline{\mathsf{Set}}$, and recall the G-tree T from Example 5.1.1. The following are the collection of (inner) faces which are *not* included in the inner horn $\Lambda^{G.c}[T]$:



The nomenclature used here is meant to be suggestive of [CM11]. Indeed, $\mathcal{F}$-∞-operads are the fibrant objects in a model structure on dSet^G:

Theorem 5.2.23 ([Per17, Theorem 2.2])**.** *For any weak indexing system $\mathcal{F}$, the category dSet^G can be endowed with the $\mathcal{F}$-model structure: a left proper cofibrantly generated model structure such that*

(1) cofibrations are the $\mathcal{F}$-normal monomorphisms;

(2) $\mathcal{F}$-anodyne extensions are trivial cofibrations;

(3) $\mathcal{F}$-∞-operads are the fibrant objects;

(4) fibrations $X \to Y$ between $\mathcal{F}$-∞-operads are inner $\mathcal{F}$-fibrations such that for all $H \leq G$, the map on categories induced by $X^H \to Y^H$ is a categorical fibration;

(5) the weak equivalences are the smallest class containing the inner $\mathcal{F}$-anodyne extensions and the trivial fibrations which is closed under 2-out-of-3.

The following conjecture would naturally extend the above result to other presheaf categories, and is currently under development

Conjecture 5.2.24. *For any weak indexing systems $\bar{\mathcal{F}}$ and $\mathcal{F}$, the category $\mathsf{dSet}_{\bar{\mathcal{F}}}$ can be endowed with an analogous $\mathcal{F}$-model structure: a left proper cofibrantly generated model structure on $\mathsf{dSet}_{\bar{\mathcal{F}}}$ such that $\mathcal{F}$-∞-operads are the fibrant objects.*

The different model structures specify different amounts of relaxing on the fixed-point rigidity found in Set^G (as opposed to $\mathsf{Set}^{O_G^{op}}$), and on the equivariance of the weak composition will be, with $\bar{\mathcal{F}} = \mathcal{F}_\varnothing$ being the most rigid, and $\bar{\mathcal{F}} = \underline{Set}$ the least. In particular, we will find that $\mathcal{F}$-∞-operads are strictifiable to genuine $\mathcal{F}$-operads.

We record some results about inner G-anodyne maps.

Lemma 5.2.25 ([Per17, Lemma 6.14], cf. [MW09, Lemma 5.1])**.** *Given $T \in \Omega_G$ and a G-closed set A of inner edges of T. Then the G-inner horn inclusion*

$$\Lambda^A[T] \to \Omega[T]$$

is inner G-anodyne.

Proof. We note that it is sufficient to show that maps of the form

$$\Lambda^E[T] \to \Lambda^{E-G.e}[T]$$

are inner anodyne. We will add all the missing faces by a series of pushouts, indexed over an equivariant poset we now describe. Picking any $e \in G.e$ chooses a tree component T_H of T and a decomposition $T \simeq G \cdot_H T_H$, where $H = \mathrm{Stab}_G(r_T)$ is the stabilizer of the root edge of T_H. We let $\mathrm{Inn}_{H.e}(T_H)$ denote the H-poset of inner faces of T_H (under inclusion) which collapse only edges in $H.e$. Thus, it suffices to check that for any H-equivariant convex subsets $B \subseteq B' \subseteq \mathrm{Inn}_{H.e}(T_H)$, we have that

$$\Lambda^E[T] \cup G \cdot_H \left(\bigcup_{T_H \setminus \bar{b} \in B} \Omega[T_H \setminus \bar{b}] \right) \to \Lambda^E[T] \cup G \cdot_H \left(\bigcup_{T_H \setminus \bar{b} \in B'} \Omega[T_H \setminus \bar{b}] \right) \tag{5.7}$$

is inner G-anodyne. We may assume that $B' = B \amalg H.(T_H \setminus \bar{b})$, so let $\bar{H} = \mathrm{Stab}_G(\bar{b})$. Now, we claim that the map 5.7 is a pushout of

$$G \cdot_{\bar{H}} \left(\Lambda^{(E \setminus G.e) \cap T_H}[T_H \setminus \bar{b}] \to \Omega[T_H \setminus \bar{b}] \right).$$

This is straightforward, once we observe that the G-stabilizer of any face not in $\Lambda^{(E \setminus G.e) \cap T_H}[T_H \setminus \bar{b}]$ is contained in $\bar{H}$; this observation follows from the fact that the set of edges $(E \setminus G.e) \cap T_H$ contains none of the conjugates of any edges $b \in \bar{b}$.

Thus, the result is proved via lexicographic induction on $(|G|, |E/G|)$. □

Lemma 5.2.26 ([Per17, Proposition 6.16], cf. [MW09, Lemma 5.2])**.** *Suppose we have a grafting of G-tree $W = T \amalg_{G.e} S$ with $G.e \simeq G/H$. Then*

$$\Omega[T] \coprod_{\Omega[G/H \cdot \eta]} \Omega[S] \to \Omega[W]$$

is inner G-anodyne.

Proof. Let $\mathrm{Out}(W)$ denote the G-poset of outer faces (i.e. non-inner non-quotient faces) of the grafted tree W, and $\mathrm{Out}_{T,S}(W)$ the G-subposet of those outer faces contained in their T nor S.

It now suffices to show that for all G-equivariant convex subsets $B \subseteq B'$ of $\mathrm{Out}_{T,S}(W)$, we have

$$\Omega[T] \amalg_{G/H} \Omega[S] \cup \bigcup_{R \in B} R \to \Omega[T] \amalg_{G/H} \Omega[S] \cup \bigcup_{R \in B'} R \tag{5.8}$$

is inner G-anodyne.

By induction, it suffices to consider the case where $B' = B \cup G.U$ for some single outer face S. Letting $H \leq G$ be the stabilizer of r_U, consider the G-tree $G \cdot_H U$, and let I_U be the set of inner edges. Then (5.8) is the pushout of

$$\Lambda^{I_S}[G \cdot_H U] \to \Omega[G \cdot_H U]$$

and the result follows via induction and the factorization

$$\Omega[T] \amalg_{G/H} \Omega[S] \to \Lambda^{I_W}[W] \to \Omega[W].$$

$\square$

Orbital Horns

We now define our second class of horn inclusions. While the underlying horns were natural generalizations via the expanded representation, as suggestively named, these will we inspired by the orbital representation of our G-trees.

Recall the poset $\Phi_{\mathrm{Orb}}(T)$ of orbital faces of T from Section 5.1.2, with examples described in Section5.1.3; in particular, they look like face maps on the orbital representation.

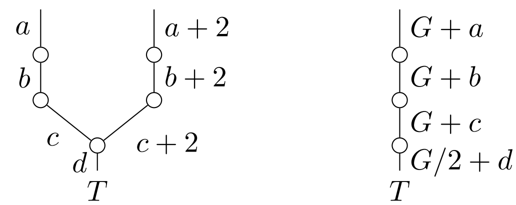
Definition 5.2.27. We define the *orbital boundary inclusion* of $\Omega[T]$ to be the map

$$\partial_{\mathrm{Orb}}\Omega[T] := \operatorname*{colim}_{\varphi \in \Phi_{\mathrm{Orb}}(T)} \partial_\varphi \Omega[T]$$

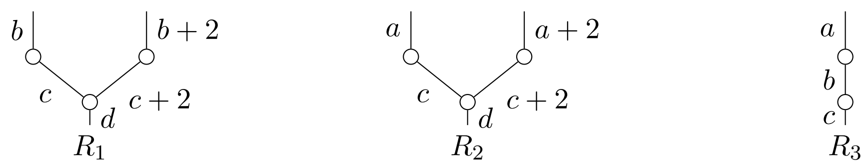
Given a G-tree T and an orbit E of inner edges, we note that any choice of element $e \in E = G.e$ induces a decomposition $T \simeq G \cdot_H T_H$, where T_H contains e. Let $\Phi^{G.e}_{\mathrm{Orb}}(T)$ be poset of orbital maps which "miss outside of $G.e$"; since $G.e$ is a single orbit, this is the equivalently the subposet of $\Phi_{\mathrm{Orb}}(T)$ excluding the orbits of faces $G \cdot (T_0 \to T)$ and $G \cdot (T_0/H.e \to T)$. Finally, define the *orbital horn inclusion* of $\Omega[T]$ to be the map

$$\Lambda^{G.e}_{\mathrm{Orb}}[T] := \operatorname*{colim}_{\varphi \in \Phi^{G.e}_{\mathrm{Orb}}(T)} \partial_\varphi \Omega[T].$$

Example 5.2.28. Let $G = \mathbb{Z}/4$, and consider the G-tree $T = C_{2G/G} \circ_G C_{G/G} \circ_{G/G} C_{G/G}$:



The orbital horn $\Lambda^{G.c}_{\mathrm{Orb}}[T]$ is generated by (the orbits of) the following non-equivariant face maps:



where

- R_1 is the orbital face associated to the outer G-face $T \setminus G.v_a$;
- R_2 is the orbital face associated to the inner G-face $T \setminus G.b$; and
- R_3 is the (one of the two) orbital face associated to the root-cluster G-face map $T \setminus G.v_d$.

A lifting diagram

$$\begin{tikzcd} \Lambda^{G.c}_{\mathrm{Orb}}[T] \arrow[r, "h"] \arrow[d, hook] & X \\ \Omega[T] \arrow[ur, dotted, "\bar{h}"'] & \end{tikzcd}$$

encodes dendrices $f_i \in X(R_i, *)$ and a lifting dendrix $\bar{h} \in i_*X(T)$ such that the inner face $\bar{\gamma} := \partial^*_{G.c}\bar{h}$ has the following compatibilities:

$$q^*(\partial^*_{G.v_a}\bar{\gamma}) = \partial^*_{\{c,c+2\}} f_1$$

$$q^*(\partial^*_{G.b}\bar{\gamma}) = \partial^*_{\{c,c+2\}} f_2$$

$$q^*(\partial^*_{G.v_d}\bar{\gamma}) = \partial^*_{E\setminus\{a,b\}} q^*\bar{\gamma} = \partial^*_{v_c} f_3$$

We will now devote the rest of this subsection to prove the following result:

Proposition 5.2.29. *For any G-tree T and any orbit of inner edges $G.e$, the orbital horn inclusion is inner G-anodyne.*

This will be necessary in order to build the homotopy strictification of a G-∞-operad as a "genuine G-operad". This begs the question as to why we use underlying horns to define G-∞-operads as opposed to these. We expect the following to hold.

Conjecture 5.2.30. *All (underlying) horn inclusions can be built cellularly out of orbital horn inclusions.*

Consequently, lifting conditions against one type hold if and only if they hold against the other. This would mean that, homotopically, they induce the same structure on dSet^G. This conjecture seems likely, especially if we interpret lifting against orbital horns as a weak "Segal-type" condition.

We will prove Proposition 5.2.29 using a "characteristic edge" argument (cf. [MW09, Lemma 9.7]), coupled with a poset induction schema (cf. [Per17]). We begin with the first piece, more abstractly.

Definition 5.2.31. Suppose we are given a decomposition $T \simeq G \cdot_H T_H$ of a G-tree T, and a subtree $U \subseteq T$, with $U_K = U \cap T_H$ and $K = Stab_H(U_K)$. Further, suppose $X \subseteq \Omega[T] \supseteq \Omega[U]$ contains all (non-equivariant) outer faces of U. We say an edge orbit $K.e \in U_K$ is a *characteristic edge orbit* if either of the two equivalent conditions hold:

(1) $\mathsf{Inn}^X[U] = G \cdot_K (\mathsf{Inn}^X_{K.e}[U_K] \times (0 \to 1)^{H.e})$; or

(2) A face $R_0 \in \mathsf{Inn}^X_e[U_K]$ if and only if $R_0/Ke \in \mathsf{Inn}^X_e[U_K]$ (equivalently, if and only if $R_0/\bar{e} \in \mathsf{Inn}^X_e[U_K]$),

where

- $\mathsf{Inn}^X[U]$ is the poset of inner faces of U not in X (or equivalently, the poset of inner edges of U not in X);

- $\mathsf{Inn}^X_{K.e}[U_K]$ is the poset of inner faces $U_K/(E \cup K.e)$ which are not in X; and

- $\mathsf{Inn}^X_e[U_K]$ is the poset of inner faces U_K/E which contain e and are not in X.

Proposition 5.2.32. *If $K.e$ is a characteristic edge orbit of $X \subseteq \Omega[T] \supseteq \Omega[U]$, then $X \to X \cup \Omega[U]$ is inner G-anodyne.*

Proof. We note that this is trivial if U (hence if $U/K.e$) are already in X. Assume otherwise. Then, it suffices to show that for all $C \subseteq C'$ K-equivariant concave subsets of $\mathrm{Inn}^X_{K.e}[U_K]$, the map

$$X \cup G \cdot_K \left(\bigcup_{E \in C} \Omega[U_K \setminus E] \right) \to X \cup G \cdot_K \left(\bigcup_{E \in C'} \Omega[U_K \setminus E] \right) \tag{5.9}$$

is inner G-anodyne; the last step would be the pushout

$$\begin{array}{ccc} G \cdot_K (\Lambda^{K.e}[U_k]) & \longrightarrow & X \cup G \cdot_K \left(\bigcup_{E \in \mathrm{Inn}^X_{K.e}} \Omega[U_K \setminus E] \right) \\ \downarrow & & \downarrow \\ G \cdot_K \Omega[U_K] \simeq \Omega[U] & \longrightarrow & X \cup \Omega[U], \end{array}$$

finishing the proof.

Now, it suffices to consider the case where C' is $C \cup K.D$, the inclusion of a single additional orbit worth of edge subsets. Without loss of generality, $e \not{\in} D$ and $U_K \setminus D$ is not in the domain. Let $\bar{K} := \mathrm{Stab}_K(D)$.

We claim that $\Lambda^{K.e}[U_K \setminus D]$ is in the domain. Indeed, if F is an outer face of $U_K \setminus D$, then F factors through an outer face of U_K, and hence is in X. Further, if $F = U_K \setminus (D \cup E)$ with $E \cap K.e = \varnothing$, concavity implies F is in the domain. Thus we are left with considering faces of the form $F = U_K \setminus D \cup \bar{e}$ with $\bar{e} \subseteq K.e$. We must show these cannot be in the domain. Suppose $U \setminus D \cup \bar{e}$ is in some $U \setminus E$ already attached; then since $E \cap K.e = \varnothing$, we have $U \setminus D \subseteq U \setminus E$, and hence $U \setminus D$ is in the domain, a contradiction. Similarly, if $U \setminus D \cup \bar{e}$ is in X, then $U \setminus D \cup K.e$ in X, and hence $U \setminus D$ in X (by definition of a characteristic orbit); so again $U \setminus D$ is in the domain, a contraction.

Thus, we are just missing precisely $\Lambda^{K.e}[U_K \setminus D]$. Further, any of the missing faces

$U \setminus D \cup \bar{e}$ also has stabilizer $\bar{K}$, or else we'd have $D \cap K.e \neq \varnothing$. Thus, we have that (5.9) is a pushout of

$$G \cdot_{\bar{K}} \left(\Lambda^{\bar{K}.e}[U_K \setminus D] \to \Omega[U_K \setminus D] \right)$$

(where this last isotropy condition ensures that the target really is freely added), and hence is anodyne, as desired. $\square$

Lemma 5.2.33. *Let U_0 be a minimal outer face of $T \simeq G \cdot_H T_H$* not *in $\Lambda^{G.e}_{\mathrm{Orb}}[T]$, and suppose $e \in U_0$. Then $K.e$ is a characteristic edge orbit for $\Lambda^{G.e}_{\mathrm{Orb}}[T] \subseteq \Omega[T] \supseteq \Omega[U_0]$*

Proof. This follows from the characterization of faces in $\Lambda^{G.e}_{\mathrm{Orb}}[T]$. Indeed, let $U_0 \setminus D$ be some inner face of U_0, with $Stab(U \setminus D) = \bar{K}$, and suppose $U \setminus (D \cup \bar{K}.e)$ is in $\Lambda^{G.e}_{\mathrm{Orb}}[T]$. Then there exists an entire L-orbit of edges in $D \cup \bar{K}.e$ away from $L.e$, and hence such an orbit exists in D away from $L.e$, where $L := \mathrm{Stab}_G(r_{U_0})$. Thus $U \setminus D \in \Lambda^{G.e}_{\mathrm{Orb}}[T]$, as required. $\square$

proof of Proposition 5.2.29. Let $\mathrm{Out}^X(T)$ be the poset of outer faces U_0 of T which are not in $\Lambda^{G.e}_{\mathrm{Orb}}[T]$. It suffices to show that for any G-convex subsets $B \subseteq B' \subseteq \mathrm{Out}^X(T)$, the map

$$\Lambda^{G.e}_{\mathrm{Orb}}[T] \cup \bigcup_{R \in B} \Omega[R] \to \Lambda^{G.e}_{\mathrm{Orb}}[T] \cup \bigcup_{R \in B'} \Omega[R']$$

is inner G-anodyne. Again, it suffices to consider the case $B' = B \cup \{U_0\}$ for some outer face U_0; without loss of generality, $e \in U_0$. Let $K = Stab_G(U_0)$.

The base case $B = \varnothing$ is given by the previous lemma. Generally, we have that

- $\Lambda^{G.e}_{\mathrm{Orb}}[T] \cup \bigcup_{R \in B} \Omega[R]$ contains all outer faces of $\Omega[U]$, by convexity; and
- $K.e$ is a characteristic edge orbit for $\Lambda^{G.e}_{\mathrm{Orb}} \cup \bigcup_{R \in B} \Omega[R] \subseteq \Omega[T] \supseteq \Omega[U]$.

Indeed, let $U \setminus D$ be an inner face of U_0 with stabilizer $\bar{K}$. Then, we have $U \setminus (D \cup \bar{K}.e)$ is in the domain if either

(1) $U \setminus D \cup \bar{K}.e \subseteq R$, for some other outer face R; but R containing an inner face of U implies R contains U, and thus, in particular, $U \setminus D \subseteq R$.

(2) $U \setminus D \cup \bar{K}e \in \Lambda^{G.e}_{\text{Orb}}[T]$, but then by the arguments in the previous lemma, $U \setminus D \in \Lambda^{G.e}_{\text{Orb}}[T]$ as well.

Finally, by Proposition 5.2.32, we have our result. □

Lastly, generalized orbital horns are also inner G-anodyne.

Lemma 5.2.34. *Let $T \in \Omega_G$ be a G-tree, and E a G-closed subset of the edges of T. Then the generalized orbital horn inclusion*

$$\Lambda^{E}_{\text{Orb}}[T] \hookrightarrow \Omega[T]$$

is inner G-anodyne.

Proof. It suffices to show that maps of the form

$$\Lambda^{E}_{\text{Orb}}[T] \hookrightarrow \Lambda^{E \setminus G.e}_{\text{Orb}}[T]$$

are inner G-anodyne. However, by observation, we note that the right-hand-side is missing precisely one generating orbital dendrix, namely $T \setminus G.b$. Thus we have a pushout

$$\begin{tikzcd}
\Lambda^{E \setminus G.b}[T \setminus G.b] \arrow[r] \arrow[d] & \Lambda^{E}_{\text{Orb}}[T] \arrow[d] \\
\Omega[T \setminus G.b] \arrow[r] & \Lambda^{E \setminus G.b}_{\text{Orb}}[T]
\end{tikzcd}$$

concluding the proof. □

5.2.3 Kan Complexes

As mentioned in the previous section, there are different strengths of inner Kan complexes. For a weak indexing system $\mathcal{F}$, let $\mathsf{Kan}_{\mathcal{F}}$ denote the full subcategory of dSet^G spanned by inner $\mathcal{F}$-Kan complexes; if $\mathcal{F} = \underline{\mathsf{Set}}$, we write Kan_G. Similarly, let $\mathsf{SKan}_{\mathcal{F}}$ denote the full subcategory of strict inner $\mathcal{F}$-Kan complexes.

We first consider the weakest notion, when $\mathcal{F} = \mathcal{F}_\varnothing$ and $\mathsf{dSet}_\mathcal{F} = \mathsf{dSet}^G$. If J denotes the set of (non-equivariant) inner horn inclusions in Ω, and $(i_G)_!J$ denotes it's image in dSet^G, we have X is a (strict) inner $\mathcal{F}_\varnothing$-Kan if and only if X has the (strict) right lifting property against $(i_G)_!J$. The following is immediate.

Lemma 5.2.35. *A presheaf $X \in \mathsf{dSet}^G$ has the (strict) right lifting property against $(i_G)_!J$ if and only if $(i_G)^*X$ has the (strict) right lifting property against J.* □

Corollary 5.2.36. $\mathsf{Kan}_{\mathcal{F}_\varnothing} = \mathsf{Kan}^G$ *and* $\mathsf{SKan}_{\mathcal{F}_\varnothing} = \mathsf{SKan}^G$. □

We will now further analyze the "strictness" condition. Equivariantly, this is a much more rigid property than its analogue in dSet, as it can be determined entirely by restricting to a subclass of inner horn inclusions, namely those into free G-trees:

Proposition 5.2.37. *If $X \in \mathsf{dSet}^G$ is a strict inner $\mathcal{F}_\varnothing$-Kan operad, then X is a strict inner G-Kan operad. That is,* $\mathsf{SKan}^G = \mathsf{SKan}_G$.

Explicitly, this is saying that if X has the strict right lifting property against all maps of the form $\Lambda^{G.e}\Omega[G \cdot T_0] \hookrightarrow \Omega[G \cdot T_0]$, then X has the strict right lifting property against all inner G-horn inclusions.

We begin the proof with a lemma:

Lemma 5.2.38. *If X is a strict inner $\mathcal{F}_\varnothing$-Kan operad, then X has strict lifts against generalized inner $\mathcal{F}_\varnothing$-horn inclusions $\Lambda^{G.A}\Omega[G \cdot T_0] \hookrightarrow \Omega[G \cdot T_0]$.*

Proof. By the proof of Lemma 5.2.25, we suspect that X must have (non-strict) lifts against these inclusions; we will show that these lifts do exist, and are unique. Given a free G-tree $G \cdot T_0$ and a subset of inner edges E of T_0, we go by induction, lexicographically on $(deg(T_0), |E|)$.

We begin by showing that X has strict lifts against inclusions of smaller generalized inner $\mathcal{F}_\varnothing$-horns into larger ones. For any $a \in A$, we have a pushout in dSet^G

$$\begin{tikzcd}
(i_G)_!\Lambda^{E\setminus e}[T_0/e] \arrow[r] \arrow[d] & (i_G)_!\Lambda^{E}[T_0] \arrow[r, "f"] \arrow[d] & X \\
(i_G)_!\Omega[T_0/e] \arrow[r] \arrow[urr, dotted, "\exists! F"] & (i_G)_!\Lambda^{E\setminus e}[T_0] \arrow[ur, dotted, "\exists! F'"'] &
\end{tikzcd}$$

For any map f, induction implies that the map F exists and is unique. By the universal property of the pushout, there is a map F' which depends uniquely on F and f, as desired. Iterating this, we find that X has unique lifts against inclusions of the form $\Lambda^{G.A}[G \cdot T_0]\hookrightarrow\Lambda^{G.A'}[G \cdot T_0]$ for any $A' \subseteq A$.

Now, suppose by induction we know that X has unique lifts against generalized inner G-horn inclusions $(i_G)_!\Lambda^E[T_0]\hookrightarrow\Omega[T_0]$ for all $|E| < n$. Given a subset of inner edges E' with $|E'| = n$, let $E = E' \setminus e$ for any choice of edge $e \in E'$; then the E'-horn inclusion factors through the E-horn inclusions. Given any $f : (i_G)_!\Lambda^{E'}[T_0] \to X$, we have the following diagram.

$$\begin{tikzcd}
(i_G)_!\Lambda^{E'}[T_0] \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, equal] \\
(i_G)_!\Lambda^{E}[T_0] \arrow[r, dotted, "\exists! g"] \arrow[d, "j"'] & X \\
(i_G)_!\Omega[T_0] \arrow[uur, dotted, "\exists \psi"] \arrow[ur, dotted, "\exists \varphi"'] &
\end{tikzcd}$$

where the dotted arrows are lifts: g is the unique lift of f against i, φ is the unique lift of g against j, and ψ is some lift of f over ji (where we note that φ is an example of such a lift). However, ψj is a lift of f over i, so by uniqueness $\psi j = g$; but then ψ is a lift of g over j, and again by uniqueness we have $\psi = \varphi$. Thus φ is the unique lift of f over ji. $\square$

proof of Proposition 5.2.37. Given a G-tree T and a horn inclusion $i : \Lambda^A[T] \to \Omega[T]$ where A is a transitive G-set of edges in T, picking any $e \in A$ yields a decomposition $T \simeq G \cdot_H T_H \simeq G \cdot T_0/N$ (where e is in the $T_0 = T_H$-th tree component) and an identification of i with $j : (i_G)_!\Lambda^{H.e}[T_0]/N\hookrightarrow(i_G)_!\Omega[T_0]/N$. Now, given $f : \Lambda^{G.e}[T] \to X$, consider the following diagram:

$$
\begin{tikzcd}
(i_G)_!\Lambda^{\{H.e\}}[T_0] \arrow[rr] \arrow[dr, "n"] \arrow[dd, "j"'] & & (i_G)\Lambda^{H.e}[T_0]/N \arrow[r, "f"] \arrow[dd, "j/N=i"] & X \\
& (i_G)_!\Lambda^{\{H.e\}}[T_0] \arrow[ur, two heads] \arrow[dd] & & \\
(i_G)_!\Omega[T_0] \arrow[rr] \arrow[dr, "n"] \arrow[rrruu, dotted, "\exists!\varphi"] & & (i_G)_!\Omega[T_0]/N & \\
& (i_G)_!\Omega[T_0] \arrow[ur, two heads] \arrow[rruuu, dotted, bend right, "\exists!\varphi"'] & &
\end{tikzcd}
$$

where n is any element N, and the dashed arrows are lifts. We note that this diagram commutes everywhere: the horn inclusion is N-equivariant, action by N is equalized in the quotient, and the lifts exist and are unique (and hence equal) by the above Lemma. Thus $\varphi = n.\varphi$ for all $n \in N$, and hence factors through the quotient; i.e. φ is a lift of f over j, and is unique, as desired. □

5.2.4 Nerves and Strictifications

In this section, "operad" will mean a G-object in the category of set operads. That is, a symmetric multicategory with a G-set of objects and appropriately equivariant sets of multimorphisms.

Recall the nerve operation from Diagram (2.4), built out of the inclusion $\Omega \xrightarrow{T \mapsto \Omega(T)} \mathsf{Op}$. Lifting to the categories of G-objects, we have the following diagram

$$
\begin{tikzcd}
\Omega^G \arrow[r] \arrow[d, "{\Omega[-]}"'] & \mathsf{Op}^G \arrow[dl, shift left, "N^G"] \\
\mathsf{dSet}^G \arrow[ur, shift left, "\tau"] &
\end{tikzcd}
$$

where $N = N^G$ is the *G-dendroidal nerve* functor, and (τ, N) is an adjoint pair. As before, if $\mathcal{P} \in \mathsf{Op}^G$, we have $N\mathcal{P}(T_G) = \mathsf{Op}^G(\Omega(T_G), \mathcal{P})$. Elements are tuples operations in $\mathcal{P}$, indexed and equivariantly colour-coordinated by the vertices of T_G.

Note that we also have a nerve-evaluation diagram out of Ω_G: the original functor $\Omega(-)$ extends to the category of forests via the coproduct of coloured operads, and again passing to G-objects and restricting along $\Omega_G \hookrightarrow \Phi^G$, we have the diagram

$$\begin{tikzcd} \Omega_G \arrow[r] \arrow[d, "{\Omega[-]}"'] & \mathsf{Op}^G \arrow[dl, shift left, "N_G"] \\ \mathsf{dSet}_G \arrow[ur, shift left, "\tau"] & \end{tikzcd}$$

However, this factors through dSet^G:

Lemma 5.2.39. *The following diagram commutes.*

$$\begin{tikzcd} & \mathsf{Op}^G \arrow[dl, "N^G"'] \arrow[d, "N_G"] \\ \mathsf{dSet}^G \arrow[r, "i_*"] & \mathsf{dSet}_G \end{tikzcd}$$

Proof. If $T = C_A$ is a G-corolla induced by an H-set A, then $N_G\mathcal{P}(C_A) \simeq \mathcal{P}(|A|)^{\Gamma_A}$, where $\Gamma_A : H \to \Sigma_{|A|}$ encodes the H-structure on A. Inductively, since the operad $\Omega(T)$ is free, we see, for general G-trees T, that

$$\begin{aligned} N_G\mathcal{P}(T) &\simeq \lim_{\mathrm{Out}_c(T)} N_G\mathcal{P}(T_v) \simeq \lim_{\mathrm{Out}_c(T)} \mathcal{P}(\#T_v)^{N_{T_v}} \\ &\simeq \left(\lim_{\mathrm{Out}_c(T)} \mathcal{P}(\#T_v)\right)^N \simeq N^G\mathcal{P}(G\cdot T_0)^N \simeq i_*N\mathcal{P}(T) \end{aligned}$$

where T_v is the G-corolla on the vertex $v \in V(T)$, and for any $T \simeq G\cdot T_0/N$, $\#T$ is the number of leaves of T_0. $\square$

Remark 5.2.40. Again, this is really the same observation that $\Phi X(G/H) = \Phi X(G/e)^H$ for the fixed-point coefficient system of a G-space — it's highlighting the fixed-point rigidity of equivariant operads, by showing that all the combinatorial information lands in the category dSet^G, which has that same rigidity, as opposed to dSet_G, which is more flexible, fixed-point wise.

Analogously to [Wei07, Theorem 3.5.12], we will show the following result:

Proposition 5.2.41. *Given $X \in \mathsf{dSet}^G$, we have that X is strict inner G-Kan operad if and only if $X \simeq N\mathcal{P}$ for some $\mathcal{P} \in \mathsf{Op}^G$.*

Remark 5.2.42. This is the strongest level of rigidity and strictness, as Lemma 5.2.20 implies that for any $\mathcal{F}$ and $\bar{\mathcal{F}}$, $N_{\bar{\mathcal{F}}}\mathcal{P}$ is a strict inner $\mathcal{F}$-Kan complex.

Proof. By Proposition 5.2.37, we can replace the first condition with $X \in \mathsf{SKan}^G$. Now, we may apply the "homotopy operad" machinery discussed in Section 2.4 by post-composing; $N_* = N : \mathsf{Op}^G \to \mathsf{dSet}^G$ as before, and new maps $\mathrm{Ho}_* : \mathsf{Kan}^G \to \mathsf{Op}^G$.

$$\begin{tikzcd} \mathsf{Op} \arrow[d, shift left, "N"] \\ \mathsf{Kan} \arrow[u, shift left, "\mathrm{Ho}(-)"] \end{tikzcd} \qquad \qquad \begin{tikzcd} \mathsf{Op}^G \arrow[d, shift left, "N_*"] \\ \mathsf{Kan}^G \arrow[u, shift left, "\mathrm{Ho}_*"] \end{tikzcd}$$

In particular, we have a natural transformation $id \Rightarrow N \circ Ho$ in the category of Kan complexes, which is an isomorphism on strict complexes; hence $X \simeq NHo_*X$ for all strict inner G-Kan complexes X.

Conversely, as observed above, elements of $N\mathcal{P}(T)$ are tuples of operations in $\mathcal{P}$, indexed and color-coordinated by the vertices of T. Since outer faces, or more specifically and relevantly the core outer poset, is always included in any inner horn of T, $N\mathcal{P}$ is clearly strict inner G-Kan. $\square$

Remark 5.2.43. This proposition could have been proved directly be rebuilding the homotopy functor on inner G-Kan complexes from [MW09]. However, that construction is no more enlightening or complicated than the above proof. In particular, $\mathrm{Ho}(X)$ can be constructed for any X with the right lifting property against just the $(i_G)_!J$, and aside from small checks that $\sim$ is equivariant (e.g. $f \sim f'$ implies $g.f \sim g.f'$ and $h \simeq f \circ_i f'$ implies $g.h \sim g.f \circ_i g.f'$), the construction and proofs go through identically as in [MW09], but now landing in Op^G.

This again demonstrates the rigidity of Op^G, and how it forces the "homotopy operad" to forgot much of the equivariant information: as any horn is the quotient of a free one, and since any two lifts (i.e. "compositions") are identified in the homotopy operad, all the extra equivariant data collapses.

Instead, we will need a new, more flexible notion of "coefficient" or "genuine equivariant operads" to allow this input to matter in the creation of the strict algebraic structure. This concept will be explored in Chapter 6.

5.3 Cofibrancy of Single-Coloured Equivariant Operads

With the technology of $\mathcal{F}$-trees, we now return to the discussion in Section 4.3. In particular, let $\mathcal{V}$ satisfy ASSUMPTION 1 (see 4.3.5), and let $\mathcal{F} = \{\mathcal{F}_n\}$ be a weak indexing system. In this section, we will investigate the properties of $\mathcal{F}$-cofibrant operads in $\mathcal{V}\mathsf{Op}^G_{\{*\}} = \mathcal{V}^G\mathsf{Op}_{\{*\}}$. As is typical, we will use the filtration built in Section 3.5 to analyze the closure properties of (trivial) cofibrations.

Remark 5.3.1. The definitions and analysis here should carry over immediately to the multicoloured case, as begun in Section 4.2.4.

We recall (Definition 4.3.2 that a map $f : \mathcal{O} \to \mathcal{P}$ is called an $\mathcal{F}$-fibration (resp. $\mathcal{F}$-weak equivalence, level $\mathcal{F}$-cofibrations) if each $f(n)$ is so in $\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$, and an $\mathcal{F}$-cofibration if it has left lifts against $\mathcal{F}$-trivial $\mathcal{F}$-fibrations.

We say $\mathcal{P} \in \mathcal{V}\mathsf{Op}^G$ is *level $\mathcal{F}$-cofibrant* if the unique map $* \to \mathcal{P}$ is a level $\mathcal{F}$-cofibration.

Remark 5.3.2. With G trivial and $\mathcal{F}_n = \{1\}$ for each n, this recovers the notion of a "Σ-cofibration" found in previous works (e.g [BM03]).

We will now prove our results. The main result says that cellular extensions from an $\mathcal{F}$-cofibrant source are $\mathcal{F}$-cofibrations.

Theorem 5.3.3. *Let $\mathcal{F}$ be a weak indexing system, and $\mathcal{V}$ a category satisfying Assumption 1. Further, let $\mathcal{P} \in \mathcal{V}\mathsf{Op}^G_{\{*\}}$ be level $\mathcal{F}$-cofibrant operad, $u : X \to Y$ a cofibration of* *symmetric sequences* $\prod \mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$*, and* $h : \mathbb{F}X \to Y$ *a map of operads. Then the cellular extension* $\mathcal{P} \to \mathcal{P}[u]$ *given by the pushout*

$$\begin{tikzcd} \mathbb{F}X \arrow[r, "h"] \arrow[d, "\mathbb{F}(u)"'] & \mathcal{P} \arrow[d] \\ \mathbb{F}Y \arrow[r] & \mathcal{P}[u] \end{tikzcd}$$

is a level $\mathcal{F}$*-cofibration, trivial if* u *is so.*

Notation 5.3.4. While we use the notion of weak indexing systems and $\mathcal{F}$-admissible trees from the above sections, the only trees we will be referring to are the usual non-equivariant objects of Ω, and hence we will denote them by just T (as opposed to T_0 used in the previous part of this chapter).

Proof. We first use the observation that $\mathcal{V}\mathsf{Op}^G_{\{*\}} = \mathcal{V}^G\mathsf{Op}_{\{*\}}$, and hence we can use our filtration from Section 3.5. Now, using the notations from Definition 3.5.9, and according to Theorem 3.5.10, it suffices to check that the map

$$\left(\mathop{\Box}_{V_a(T)} \iota_{P(T_v)} \,\Box\, [u]^{\Box V_p(T)} \right) \otimes_{\mathrm{Aut}(T)} \Sigma_n$$

is a (trivial) cofibration in $\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$ given our assumptions.

By Lemma 5.3.7,

$$\Sigma_n \otimes_{\mathrm{Aut}(T)} (-) : \mathcal{V}^{G\times\mathrm{Aut}(T)}_{\mathcal{F}_T} \to \mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$$

is left Quillen if and only if $val(\Gamma) \in \mathcal{F}_n$ whenever $\Gamma \leq G \times \mathrm{Aut}(T)$ defines an $\mathcal{F}$-admissible tree with n leaves. Thus, we see this holds precisely when $\mathcal{F}$ is a weak indexing system.

It remains to show that the given map in $\mathcal{V}^{G\times\mathrm{Aut}(T)}_{\mathcal{F}_T}$ is a (trivial) cofibration. We observe that this map is a large indexed box product, in particular over a list of (trivial) cofibrations $f(v) \in \mathcal{V}^{G\times\Sigma_{T_v}}_{\mathcal{F}_{T_v}}$ such that $f(v) = f(\alpha(v))$ for all $v \in V(T)$ and $\alpha \in \mathrm{Aut}(T)$. The fact that this is again a (trivial) cofibration is the content of Proposition 5.3.27. $\square$

Corollary 5.3.5 (cf. [BM03, Corollary 5.2, Proposition 4.3])**.** *The class of level $\mathcal{F}$-cofibrant operads in $\mathcal{V}\mathsf{Op}^G_{\{*\}}$ is closed under cellular extensions. Moreover, if $\mathcal{O} \in \mathcal{V}\mathsf{Op}^G$ is $\mathcal{F}$-cofibrant, then the underlying symmetric sequence is level cofibrant.*

Proof. The main result is an immediate consequence of Theorem 5.3.3. For the moreover, we recall that any $\mathcal{F}$-cofibrant operad $\mathcal{O}$ can be built out of a retract of a composite of cellular extensions of generating $\mathcal{F}$-cofibrations $u : X \to Y$ of symmetric sequences, starting with the initial operad. As the initial operad is just the initial object of $\mathcal{V}$ in each level, it is level $\mathcal{F}$-cofibrant for all $\mathcal{F}$. Thus, the moreover follows from the main result and the observation that level $\mathcal{F}$-cofibrations are closed under retracts. $\square$

We spend the rest of this section proving Proposition 5.3.27. This will follow from a discussion of the interplay between different families, symmetric products, and trees.

Remark 5.3.6. The outline of the following story, in particular a description of the types of constructions needed, was conveyed to me by Pereira.

We first note some basic relations.

Lemma 5.3.7. *Suppose we are given two groups Π and $\bar{\Pi}$, families of subgroups $\mathcal{F}$ and $\bar{F}$, and a homomorphism $\varphi : \Pi \to \bar{\Pi}$ of groups. Then the induced adjunction*

$$\mathsf{fgt} : \mathcal{V}^{\bar{\Pi}}_{\bar{\mathcal{F}}} \leftrightarrows \mathcal{V}^{\Pi}_{\mathcal{F}} : \bar{\Pi} \cdot_{\Pi} (-)$$

is a Quillen pair if and only if for all $H \in \mathcal{F}$, $\varphi(H) \in \bar{\mathcal{F}}$.

Proof. We have that $\bar{\Pi} \cdot_{\Pi} (\Pi/H) = \bar{\Pi}/\varphi(H)$, and thus under precisely these conditions do generating cofibrations $\Pi/H \cdot i)$ in $\mathcal{V}^{\Pi}_{\mathcal{F}}$ get sent to generating cofibrations in $\mathcal{V}^{\bar{\Pi}}_{\bar{\mathcal{F}}}$. $\square$

Dually:

Lemma 5.3.8. *Given φ as above, the induced adjunction*

$$Hom^{\Pi}(\bar{\Pi}, -) : \mathcal{V}^{\Pi}_{\mathcal{F}} \leftrightarrows \mathcal{V}^{\bar{\Pi}}_{\bar{\mathcal{F}}} : \mathsf{fgt}$$

is a Quillen pair if and only if for all $\bar{H} \in \bar{F}$, we have $\varphi^{-1}(\bar{H}^g) \in \mathcal{F}$.

Proof. The generalized double coset formula tells us that

$$\mathsf{res}_{\Pi}^{\bar{\Pi}}(\bar{\Pi}/\bar{H}) = \coprod_{g \in \varphi(H)\backslash\bar{\Pi}/\bar{H}} \Pi/\varphi^{-1}(\bar{H}^g),$$

and so again the result follows immediately. □

The results above lead us to create the following definitions:

Definition 5.3.9. Given a homomorphism $\varphi : \Pi \to \bar{\Pi}$, and families $\mathcal{F}$ and $\bar{\mathcal{F}}$, we define

$$\varphi^*(\bar{\mathcal{F}}) = \left\{H \leq \Pi \mid \varphi(H) \in \bar{\mathcal{F}}\right\} \tag{5.10}$$

$$\varphi_!(\mathcal{F}) = \left\{\varphi(H)^{\bar{g}} \leq \bar{\Pi} \mid \bar{g} \in \bar{\Pi}, H \in \mathcal{F}\right\} \tag{5.11}$$

$$\varphi_*(\mathcal{F}) = \left\{\bar{H} \leq \bar{\Pi} \mid \varphi^{-1}(\bar{H}^{\bar{g}}) \in \mathcal{F} \text{ for all } \bar{g} \in \bar{\Pi}.\right\} \tag{5.12}$$

Remark 5.3.10. We see the results of Lemma 5.3.7 hold if and only if $\mathcal{F} \subseteq \varphi^*(\bar{\mathcal{F}})$ if and only if $\varphi_!(\mathcal{F}) \subseteq \bar{\mathcal{F}}$; dually, the results of Lemma 5.3.8 hold if and only if $\bar{F} \subseteq \varphi_*(F)$ if and only if $\varphi^*(\bar{\mathcal{F}}) \subseteq \mathcal{F}$.

Given multiple families of subgroups, we can combine them in different ways for form new families. First, if $\mathcal{F}$ and $\bar{\mathcal{F}}$ are both families of subgroups of the same group Π, we let $\mathcal{F} \cap \bar{\mathcal{F}}$ denote their *internal intersection*, their intersection as sets.

Lemma 5.3.11. *Let $\mathcal{F}$ and $\bar{\mathcal{F}}$ be families of subgroups of Π; then*

$$\otimes : \mathcal{V}_{\mathcal{F}}^{\Pi} \times \mathcal{V}_{\bar{\mathcal{F}}}^{\Pi} \to \mathcal{V}_{\mathcal{F}\cap\bar{\mathcal{F}}}^{\Pi}$$

is left Quillen.

Proof. This again follows from the double coset formula, the description of generating cofibrations, and the fact that families are closed under subgroups:

$$(\Pi/H \cdot i)\square(\bar{\Pi}/\bar{H} \cdot \bar{i}) = (\Pi/H \times \bar{\Pi}/\bar{H}) \cdot (i\square\bar{i}) = \left(\coprod \Pi/H \cap \bar{H}^g\right) \cdot (i\square\bar{i}).$$

□

Second, if $\mathcal{F}$ and $\bar{\mathcal{F}}$ are families of subgroups of *different* groups Π and $\bar{\Pi}$, respectively, we define their *external intersection*, denoted $\mathcal{F} \sqcap \bar{\mathcal{F}}$, to be their internal intersection in $\Pi \times \bar{\Pi}$ after pulling back:

$$\mathcal{F} \sqcap \bar{\mathcal{F}} := \pi^*\mathcal{F} \cap \bar{\pi}^*\bar{\mathcal{F}},$$

where $\pi : \Pi \times \bar{\Pi} \to \Pi$ and $\bar{\pi} : \Pi \times \bar{\Pi} \to \bar{\Pi}$ are the projections.

Lemma 5.3.12. *The natural map*

$$\otimes : \mathcal{V}^{\Pi}_{\mathcal{F}} \times \mathcal{V}^{\bar{\Pi}}_{\bar{\mathcal{F}}} \to \mathcal{V}^{\Pi\times\bar{\pi}}_{\mathcal{F}\sqcap\bar{\mathcal{F}}}$$

is left Quillen.

Proof. This follows from Lemmas 5.3.8 and 5.3.11 via the composite

$$\mathcal{V}^{\Pi}_{\mathcal{F}} \times \mathcal{V}^{\bar{\Pi}}_{\bar{\mathcal{F}}} \xrightarrow{\mathsf{fgt}} \mathcal{V}^{\Pi\times\bar{\Pi}}_{\pi_1^*\mathcal{F}} \times \mathcal{V}^{\Pi\times\bar{\Pi}}_{\pi_2^*\bar{\mathcal{F}}} \xrightarrow{\otimes} \mathcal{V}^{\Pi\times\bar{\Pi}}_{\mathcal{F}\sqcap\bar{\mathcal{F}}}.$$

□

5.3.1 Box Products

Given a family $\mathcal{F}$ of subgroups of Π, we would like to construct a natural family $\mathcal{F}^{\otimes k}$ of subgroups of $\Sigma_k \wr \Pi$ for any $k \in \mathbb{N}$. In particular, we would like the family so that $H \in \mathcal{F}^{\otimes k}$ only if the projection of H onto each of the k copies of Π is admissible. However, the set function

$$\Sigma_k \wr \Pi = \Sigma_k \ltimes \Pi^{\times k} \xrightarrow{\pi_i} \Pi$$

is *not* a group homomorphism. It is, however, if we restrict to $(\Sigma_{\{i\}} \times \Sigma_{k\setminus\{i\}}) \ltimes \Pi^{\times k}$. So, we will first pullback $\mathcal{F}$ along this map, and then push-forward the resultant along the inclusion into $\Sigma_k \wr \Pi$.

Definition 5.3.13. Given a family $\mathcal{F}$ of subgroups of Π, define $\mathcal{F}^{\otimes k}$ to be the family of subgroups given by

$$\mathcal{F}^{\otimes k} = \bigcap_{i=1}^{k} (\omega_i)_* \pi_i^*(\mathcal{F}),$$

where

$$\pi_i : (\Sigma_{\{i\}} \times \Sigma_{k\setminus\{i\}}) \ltimes \Pi^k \to \Pi$$

projects onto the i-th Π coordinate, and

$$\omega_i : (\Sigma_{\{i\}} \times \Sigma_{k\setminus\{i\}}) \ltimes \Pi^k \hookrightarrow \Sigma_k \wr \Pi$$

is the inclusion.

Unpacking definitions, this says that $H \in \mathcal{F}^{\otimes k}$ if and only if

$$\pi_i \left(H^g \cap \left((\Sigma_{\{i\}} \times \Sigma_{k\setminus\{i\}}) \ltimes \Pi^k \right) \right) \in \mathcal{F}$$

for every $g \in \Pi$ and $i \in \{1, \ldots, k\}$. However, it is clear that this family is over-defined. In particular, if $\tau_{ii'} = ((ii'), (id)) \in \Sigma_k \wr \Pi$, then

$$\pi_{i'} \left(H^g \cap \left((\Sigma_{\{i'\}} \times \Sigma_{k\setminus\{i'\}}) \ltimes \Pi^k \right) \right) = \pi_i \left(H^{g \circ \tau_{ii'}} \cap \left((\Sigma_{\{i\}} \times \Sigma_{k\setminus\{i\}}) \ltimes \Pi^k \right) \right).$$

Hence:

Lemma 5.3.14. *For any $i, i' \in \{1, \ldots, k\}$, we have*

$$(\omega_{i'})_* \pi_{i'}^* \mathcal{F} = (\omega_i)_* \pi_i^* \mathcal{F}.$$

□

Moreover, if $g = (\tau, (g_j)) \in \Sigma_k \wr \Pi$ such that $\tau(i') = i$, then we determine that

$$\pi_i \left(H^g \cap \left((\Sigma_{\{i\}} \times \Sigma_{k \setminus \{i\}}) \ltimes \Pi^k \right) \right) = g_i \pi_{i'} \left(H \cap \left((\Sigma_{\{i'\}} \times \Sigma_{k \setminus \{i'\}}) \ltimes \Pi^k \right) \right) g_i^{-1},$$

using the observation that if σ fixes i, then $\tau\sigma\tau^{-1}$ (τ as above) fixes i'. Thus, we have an equivalent characterization of $\mathcal{F}^{\otimes k}$:

Lemma 5.3.15. *$H \in \mathcal{F}^{\otimes k}$ if and only if $\pi_i \left(H \cap \left((\Sigma_{\{i\}} \times \Sigma_{k \setminus \{i\}}) \ltimes \Pi^k \right) \right) \in \mathcal{F}$ for all $i \in \{1, \ldots, k\}$.* □

The following will allow us to use the machinery of [Per16] to prove Proposition 5.3.17.

Lemma 5.3.16. *For any family $\mathcal{F}$ of subgroups of Π, we have*

$$\mathcal{F}^{\otimes k_1} \sqcap \mathcal{F}^{\otimes k_2} \subseteq \omega^* \left(\mathcal{F}^{\otimes (k_1 + k_2)} \right),$$

where ω is the inclusion

$$\omega : \Sigma_{k_1} \wr \Pi \times \Sigma_{k_2} \wr \Pi = (\Sigma_{k_1} \times \Sigma_{k_2}) \wr \Pi \to \Sigma_{k_1 + k_2} \wr \Pi.$$

Proof. By definition of $\sqcap$, we see that $H \leq \Sigma_{k_1} \wr \Pi \times \Sigma_{k_2} \wr \Pi = (\Sigma_{k_1} \times \Sigma_{k_2}) \wr \Pi$ is in $\mathcal{F}^{\otimes k_1} \sqcap \mathcal{F}^{\otimes k_2}$ if and only if

$$\begin{aligned} \pi_i \left(H \cap \left((\Sigma_{\{i\}} \times \Sigma_{k_1 \setminus \{i\}} \times \Sigma_{k_2}) \ltimes \Pi^{k_1 + k_2} \right) \right) &\in \mathcal{F} \qquad \text{for all } i \in \{1, \ldots, k_1\} \\ \pi_{i'} \left(H \cap \left((\Sigma_{k_1} \times \Sigma_{\{i'\}} \times \Sigma_{k_2 \setminus \{i'\}}) \ltimes \Pi^{k_1 + k_2} \right) \right) &\in \mathcal{F} \qquad \text{for all } i' \in \{k_1 + 1, \ldots, k_1 + k_1\}. \end{aligned}$$

Moreover, for any $j \in \{1, \ldots, k_1 + k_2\}$, $\omega(H) \cap \left((\Sigma_{\{j\}} \times \Sigma_{k_1 + k_2 \setminus \{j\}}) \ltimes \Pi^{k_1 + k_2} \right)$ is equal to

$$\begin{aligned} &H \cap \left((\Sigma_{\{j\}} \times \Sigma_{k_1 \setminus \{j\}} \times \Sigma_{k_2}) \ltimes \Pi^{k_1 + k_2} \right) \qquad \text{if } j \in \{1, \ldots, k_1\}, \text{ and} \\ &H \cap \left((\Sigma_{k_1} \times \Sigma_{\{j\}} \times \Sigma_{k_2 \setminus \{j\}}) \ltimes \Pi^{k_1 + k_2} \right) \qquad \text{if } j \in \{k_1 + 1, \ldots, k_1 + k_2\}. \end{aligned}$$

Hence $\varphi(H) \in \mathcal{F}^{\otimes k_1+k_2}$, and the result follows. $\square$

We will now prove the following analogue of [Per16, Theorem 1.2].

Proposition 5.3.17. *If f is a (trivial) cofibration in $\mathcal{V}^{\Pi}_{\mathcal{F}}$, then so is $f^{\square k} \in \mathcal{V}^{\Sigma_k \wr \Pi}_{\mathcal{F}^{\otimes k}}$.*

The technology used it that proof can be applied here, using Lemmas 5.3.16 and 5.3.7. We break down the results into lemmas themselves.

Notation 5.3.18. We recall a simplification of the notation in 3.5.9. Given a convex subposet $\mathcal{C}$ of $(0 \to 1)^{\times k}$ and a diagram $u : A_0 \to A_1$ in $\mathcal{V}^{(0\to 1)}$, define $Q^k_{\mathcal{C}}[u] := \operatorname{colim}_{\mathcal{C}}[u]$.

In particular, let $\mathcal{C}_t$ be the subposet of $(0 \to 1)^{\times k}$ of tuples with at most t 1-entries, and $Q^k_t[u] := Q^k_{\mathcal{C}_t}[u]$.

More generally, given a poset $\mathcal{D}$, a convex subposet $\mathcal{C}$ of $\mathcal{D}^{\times k}$, and a diagram $i : \mathcal{D} \to \mathcal{V}$, define $Q^n_{\mathcal{C}}[i] := \operatorname{colim}_{\mathcal{C}}(i^{\otimes k})$.

In particular, given $\bar{e} \in \mathcal{D}^{\times k}$, let $\mathcal{C}_{\bar{e}} := \left\{\bar{e}' \in \mathcal{D}^{\times n} \mid \bar{e}' < \bar{e}\right\}$. This comes with natural latching maps $\lambda^k_{\bar{e}}(i) : Q^k_{\mathcal{C}_{\bar{e}}}[i] \to i^{\otimes k}(\bar{e})$.

Lemma 5.3.19. *[cf. [Per16, Lemma 4.8]] Letting $\mathcal{D}$ denote the diagram category $(0 \to 1 \to 2)$, and $d : \mathcal{D} \to \mathcal{V}^{\Pi}$ the diagram*

$$Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2.$$

Suppose that $f_i^{\square k} : Q^k[f_i] \to Z_i^{\otimes k}$ is a (trivial) cofibration in $\mathcal{V}^{\Sigma_k \wr \Pi}_{\mathcal{F}^{\otimes k}}$ for $i \in \{0, 1\}$. Then, for all $\mathcal{C} \subseteq \mathcal{C}'$ symmetric convex subposets of $(0 \to 1\ to 2)^{\times n}$ which contain all tuples with at least one 0-entry, the map

$$Q^k_{\mathcal{C}}[i] \to Q^k_{\mathcal{C}'}[i]$$

is a (trivial) cofibration in $\mathcal{V}^{\Sigma_k \wr \Pi}_{\mathcal{F}^{\otimes k}}$.

Proof. Without loss of generality, $\mathcal{C}' = \mathcal{C} \cup \Sigma_n.e$ for some tuple $\bar{e} = e_o \amalg e_1 \amalg e_2$ with k_0

0-entries, k_1 1-entries, and k_2 2-entries. We note that $\mathcal{C}_{\bar{e}} \subseteq \mathcal{C}$, and hence we have a pushout

$$\begin{tikzcd}
\Sigma_k \cdot_{\Sigma_{k_0}\times\Sigma_{k_1}\times\Sigma_{k_2}} Q^k_{\mathcal{C}_{\bar{e}}}[i] \arrow[r] \arrow[d, "\Sigma_k\cdot_{\Sigma_{k_0}\times\Sigma_{k_1}\times\Sigma_{k_2}}\lambda^k_{\bar{e}}[i]"'] & Q^k_{\mathcal{C}}[i] \arrow[d] \\
\Sigma_k \cdot_{\Sigma_{k_0}\times\Sigma_{k_1}\times\Sigma_{k_2}} Z_0^{\otimes k_0} \otimes Z_1^{\otimes k_1} \otimes Z_2^{\otimes k_2} \arrow[r] & Q^k_{\mathcal{C}'}[i].
\end{tikzcd}$$

Lemmas 5.3.16 and 5.3.7 imply it suffices to show that $\lambda^k_{\bar{e}}[i]$ is a cofibration in $\mathcal{V}^{\Sigma_{k_0}\times\Sigma_{k_1}\times\Sigma_{k_2}\wr\Pi}_{\mathcal{F}^{\otimes k_0}\sqcap\mathcal{F}^{\otimes k_1}\sqcap\mathcal{F}^{\otimes k_2}}$. This follows by observing that this latching map has a decomposition

$$\lambda^k_{\bar{e}}[i] = \lambda^{k_0}_{\bar{e}_0}[i]\Box\lambda^{k_1}_{\bar{e}_1}[i]\Box\lambda^{k_2}_{\bar{e}_2}[i] = Z_0^{\otimes k_0} \otimes (f_1^{\Box k_1}\Box f_2^{\Box k_2}).$$

By assumption on $\mathcal{C}$ and $\mathcal{C}'$, we have that $k_0 = 0$, and hence an application of Lemma 5.3.12 finished the proof. $\Box$

Corollary 5.3.20. *[cf. [Per16, Lemma 4.10]] Let $Z_0 \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2$ be as above. Then*

$$Q^k_{k-1}[f_2 f_1] \coprod_{Q^k_{k-1}[f_1]} Z_1^{\otimes k} \to Z_2^{\otimes k}$$

is a cofibration in $\mathcal{V}^{\Sigma_k\wr\Pi}_{\mathcal{F}^{\otimes k}}$.

Proof. This immediately follows from the above by identifying each as a model of $Q^k_{\mathcal{C}}[i]$:

- $Q^k_{k-1}[f_1] = Q^k_{\mathcal{C}^1_{k-1}}[u]$ with $\mathcal{C}^1_{k-1}$ the set of tuples with no 2-entries and at most $(k-1)$ 1-entires;
- $Q^k_{k-1}[f_2 f_1] = Q^k_{\mathcal{C}^2_{k-1}}[u]$ with $\mathcal{C}^2_{k-1}$ the set of tuples with at most $k-1$ 1- or 2-entries.

$\Box$

proof of Proposition 5.3.17. This is immediate on generators, as

$$(\Pi/H \cdot i)^{\Box k} = (\Pi/H)^{\times k} \cdot i^{\Box k} \simeq (\Sigma_k \wr \Pi/\Sigma_k \wr H) \cdot i^{\Box k},$$

and $\Sigma_k \wr H$ is clearly in $\mathcal{F}^{\otimes k}$.

Given a general trivial cofibration f, we write

$$f : A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \to \dots \to A_\kappa = \operatorname*{colim}_{\beta<\kappa} A_\beta,$$

with

- $f_\beta : A_\beta \to A_{\beta+1}$ the pushout of a generating cofibration i_β of $\mathcal{V}^{\Pi}_{\mathcal{F}}$;
- $A_\beta = \mathrm{colim}_{\gamma<\beta} A_\gamma$ for any limit ordinals $\beta < \kappa$; and
- $\bar{f}_\beta : A_0 \to A_\beta$ the composition for any $\beta < \kappa$.

Since $Q^k_{k-1}[-]$ and $\otimes$ preserve filtered colimits, it suffices to show that the vertical maps below are cofibrations in $\mathcal{V}^{\Sigma_k \wr \Pi}_{\mathcal{F}^{\otimes k}}$;

$$\begin{tikzcd}
Q^k_{k-1}[f_0] \arrow[r] \arrow[d] & Q^k_{k-1}[f_1 f_0] \arrow[r] \arrow[d] & Q^k_{k-1}[f_2 f_1 f_0] \arrow[r] \arrow[d] & \dots \arrow[r] & Q^k_{k-1}[\bar{f}_\kappa] \arrow[d, dotted] \\
A_1^{\otimes k} \arrow[r] & A_2^{\otimes k} \arrow[r] & A_3^{\otimes k} \arrow[r] & \dots \arrow[r] & A_\kappa^{\otimes k}
\end{tikzcd}$$

By induction, it suffices to check that the *relative* latching maps

$$A_\beta^{\otimes k} \coprod_{Q^k_{k-1}[\bar{f}_\beta]} Q^k_{k-1}[\bar{f}_{\beta+1}] \to A_{\beta+1}^{\otimes k}$$

are cofibrations in $\mathcal{V}^{\Sigma_k \wr \Pi}_{\mathcal{F}^{\otimes k}}$; the base case has $Q^k_{k-1}[\bar{f}_0] = A_0^{\otimes k}$ and hence is trivial, as is the case for any limit ordinal by our convention above. Now, by assumption, $\bar{f}_\beta^{\square k}$ is a (trivial) cofibration, while Lemma 5.3.19 (applied to i_β and f_β) and the case for generating (trivial) cofibrations implies that $f_\beta^{\square k}$ is a (trivial) cofibration. Thus, applying Lemma 5.3.20 $A_0 \xrightarrow{\bar{f}_\beta} A_\beta \xrightarrow{f_0} A_{\beta+1}$ yields that the latching map is indeed a (trivial) cofibration, as desired. $\square$

5.3.2 Graph Subgroups

When are families are now specifically families of graph subgroups of $G \times \Pi$, our analysis needs to be twisted accordingly.

Definition 5.3.21. Given $\mathcal{F}$ and $\bar{\mathcal{F}}$ families of graph subgroups of $G \times \Pi$ and $G \times \bar{\Pi}$ respectively, we define the G-*external intersection*, denoted $\mathcal{F} \sqcap_G \bar{\mathcal{F}}$, as family of subgroups of $G \times \Pi \times \bar{\Pi}$ given by

$$\mathcal{F} \sqcap_G \bar{\mathcal{F}} := \Delta^*(\mathcal{F} \sqcap \bar{\mathcal{F}}),$$

where $\Delta : G \times \Pi \times \bar{\Pi} \to G \times \Pi \times G \times \bar{\Pi}$ is the diagonal map.

Equivalently, we see that

$$\mathcal{F} \sqcap_G \bar{\mathcal{F}} = \left\{ \Gamma(\varphi \times \bar{\varphi}) \mid G \hookleftarrow H \xrightarrow{\varphi \times \bar{\varphi}} \Pi \times \bar{\Pi},\ \Gamma(\varphi) \in \mathcal{F},\ \Gamma(\bar{\varphi}) \in \bar{\mathcal{F}} \right\}.$$

Lemma 5.3.22. *$\mathcal{F} \sqcap_G \bar{\mathcal{F}}$ is a family of graph subgroups of $G \times \Pi \times \bar{\Pi}$.*

Proof. Both subgroups and conjugation preserve the product structure; hence this follows since $\mathcal{F}$ and $\bar{\mathcal{F}}$ are families of graph subgroups. □

Lemma 5.3.12 extends to this context.

Lemma 5.3.23. *Given $\mathcal{F}$ and $\bar{\mathcal{F}}$ families of graph subgroups of $G \times \Pi$ and $G \times \bar{\Pi}$ respectively, the map*

$$\mathcal{V}^{G\times\Pi}_{\mathcal{F}} \times \mathcal{V}^{G\times\bar{\Pi}}_{\bar{\mathcal{F}}} \xrightarrow{\otimes} \mathcal{V}^{G\times\Pi\times\bar{\Pi}}_{\mathcal{F}\sqcap_G\bar{\mathcal{F}}}$$

is left Quillen.

Proof. This also follows by Lemmas 5.3.8 5.3.11, via the factorization

$$\mathcal{V}^{G\times\Pi}_{\mathcal{F}} \times \mathcal{V}^{G\times\bar{\Pi}}_{\bar{\mathcal{F}}} \xrightarrow{\mathsf{fgt}} \mathcal{V}^{G\times\Pi\times G\times\bar{\Pi}}_{\pi^*\mathcal{F}} \times \mathcal{V}^{G\times\Pi\times G\times\bar{\Pi}}_{\pi^*\bar{\mathcal{F}}} \xrightarrow{\otimes} \mathcal{V}^{G\times\Pi\times G\times\bar{\Pi}}_{\mathcal{F}\sqcap\bar{\mathcal{F}}} \xrightarrow{\mathsf{fgt}} \mathcal{V}^{G\times\Pi\times\bar{\Pi}}_{\mathcal{F}\sqcap_G\bar{\mathcal{F}}}.$$

□

We will also twist the families $\mathcal{F}^{\otimes k}$ to account for the special purpose of G.

Definition 5.3.24. Given a family $\mathcal{F}$ of graph subgroups of $G \times \Pi$, we define $\mathcal{F}^{\otimes_G k}$ to be the family of subgroups of $G \times \Sigma_k \wr \Pi$ given by

$$\Delta^* \left(\mathcal{F}^{\otimes k}\right),$$

where $\Delta : G \times \Sigma_k \wr \Pi \to \Sigma_k \wr (G \times \Pi)$ is the diagonal map on G.

Proposition 5.3.17 extends to this context.

Proposition 5.3.25. *Let $\mathcal{F}$ be a family of graph subgroups of $G \times \Pi$. If f is (trivial) cofibration in $\mathcal{V}^{G\times\Pi}_{\mathcal{F}}$, then so is $f^{\square n}$ in $\mathcal{V}^{G\times\Sigma_n\wr\Pi}_{\mathcal{F}^{\ltimes_G n}}$.*

Proof. This follows from a single application of Lemma 5.3.8 along the map

$$\mathcal{V}^{\Sigma_k\wr(G\times\Pi)}_{\mathcal{F}^{\otimes k}} \xrightarrow{\mathsf{fgt}} \mathcal{V}^{G\times\Sigma_k\wr\Pi}_{\mathcal{F}^{\otimes_G k}}.$$

$\square$

5.3.3 Tree Families

We now provide a description of the family of graph subgroups of $\mathcal{F}$-admissible trees. Given a tree T, let

$$\pi_{G\times\Sigma_m} = G \times \pi_{C_m} : G \times \mathrm{Aut}(T) \to G \times \Sigma_m$$
$$\pi_i = \pi_{G\times T_i} = G \times \pi_{T_i} : G \times \mathrm{Aut}(T) \to G \times \mathrm{Aut}(T_i)$$

denote the restrictions to the root corolla and the i-th branch component, respectively.

Lemma 5.3.26. *For an collection of families $\mathcal{F} = \{\mathcal{F}_n\}$, and $T \in \Omega$ not a corolla, we have*

$$\mathcal{F}_T \cong (\pi_{G\times\Sigma_m})^*(\mathcal{F}_m) \ \cap \ \left(\mathcal{F}^{\otimes_G k_1}_{T_{i_1}} \sqcap_G \dots \sqcap_G \mathcal{F}^{\otimes_G k_r}_{T_{i_r}}\right),$$

where $T \simeq C_m(T_1, \dots, T_m) = C_m(T_1^1, \dots, T_1^{k_1}, T_2^1, \dots, T_r^{k_r})$ *is a grafting decomposition of* T.

Proof. Let $\bar{\mathcal{F}}_T$ denote the right-hand side. Unpacking definitions, we see that $\Gamma \in \bar{\mathcal{F}}_T$ if, for all $i \in \{1, \dots, m\}$, we have

$$\pi_i \left(\pi_{G \times \Sigma_{k_i} \wr \mathrm{Aut}(T_i)} \Gamma \cap \left(G \times (\Sigma_i \times \Sigma_{k_{j_i}}) \ltimes \mathrm{Aut}(T_i)^{k_{j_i}} \right) \right) \in \mathcal{F}_{T_i}.$$

However, by just including the rest of $\mathrm{Aut}(T)$, we have this is equivalent to the condition

$$\pi_i \left(\Gamma \cap \left(G \times (\Sigma_i \times \Sigma_{k_{j_i}}) \ltimes \mathrm{Aut}(T_i)^{k_{j_i}} \times \prod_{j \neq j_i} \Sigma_{k_j} \wr \mathrm{Aut}(T_j) \right) \right) \in \mathcal{F}_{T_i}.$$

Moreover, in this expanded representation, it is clear that what π_i is pushing forward is in fact precisely

$$\Gamma \cap \left(G \times \pi_{C_m}^{-1} (\Sigma_{\{i\}} \times \Sigma_{n \setminus \{i\}}) \right).$$

Now, let us assume that Γ is a graph subgroup $\Gamma(\alpha)$ of $G \times \mathrm{Aut}(T)$. Then $\Gamma \cap (G \times \pi_{C_m}^{-1}(\Sigma_{\{i\}} \times \Sigma_{n \setminus \{i\}}))$ is precisely $\Gamma(\alpha|_{H_i})$, where $H_i = \mathrm{Stab}(e_i)$ is the stabilizer of the root of T_i under the action on T induced by Γ; we further note that $\pi_i(\Gamma|_{H_i}) = \Gamma|_{T_i}$ is the graph subgroup describing the induced action on the branch T_i. Similarly, $\pi_{C_m}(\Gamma) = \Gamma|_{T_{v_r}}$ is the graph subgroup describing the induced action on the root corolla T_{v_r}. Thus, with this graph subgroup assumption, we have

$$\begin{aligned}
\Gamma \in \bar{\mathcal{F}}_T &\Leftrightarrow \Gamma|_{T_i} \in \mathcal{F}_{T_i} \text{ for all } i \in \{1, \dots, m\} \text{ and } \Gamma|_{T_{v_r}} \in \mathcal{F}_n \\
&\Leftrightarrow \Gamma|_{T_i} \text{ and } \Gamma|_{T_{v_r}} \text{ induce } \mathcal{F}\text{-admissible structures on the } T_i \text{ for all } i \text{ and on the root corolla} \\
&\Leftrightarrow \Gamma \in \mathcal{F}_T.
\end{aligned}$$

Thus, it suffices to show that all $\Gamma \in \bar{\mathcal{F}}_T$ are in fact graph subgroups. That is, given $(1, \sigma) \in \Gamma$, we claim σ is also the identity. Indeed, since $\pi_{G \times \Sigma_m}^*(\Gamma) \in \mathcal{F}_m$, any $\gamma \in \Gamma$ that is of the form $(1,\sigma)$ must have σ act as the identity on Σ_m. This implies $\sigma = \prod \sigma_i$ for some $\sigma_i \in \mathrm{Aut}(T_i)$. But then $(1,\sigma_i) \in \pi_i\Gamma \in \mathcal{F}_{T_i}$ implies that these σ_i must also be the identity, as desired. □

Now, we finally have all the pieces to prove the key technical result of this section.

Proposition 5.3.27. *Let $\mathcal{F}$ be a collection of families (not necessarily a weak indexing system), and $T \in \Omega$. Further, suppose we are given (trivial) cofibrations $f(v) \in \mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$ for each $v \in V(T)$, such that $f(v) = f(\alpha(v))$ for all $\alpha \in \mathrm{Aut}(T)$. Then the box product*

$$f^{\square V(T)} = \mathop{\square}_{v\in V(T)} f(v)$$

is a (trivial) cofibration in $\mathcal{V}^{G\times \mathrm{Aut}(T)}_{\mathcal{F}_T}$.

Proof. Using the grafting decomposition $T \simeq t_m(T_1^1, \ldots, T_1^{k_1}, T_2^1, \ldots, T_r^{k_r})$, we go by induction on the number of vertices of T. The base case of $|V(T)| = 0$ or 1 is trivial. Now, we note that we have a decomposition

$$f^{\square V(T)} = f(v_r) \square \mathop{\square}_{i=1}^{r} \left(\left(f^{\square V(T_i)} \right)^{\square k_i} \right).$$

We build this map progressively, preserving (trivial) cofibrancy in each step.

- By induction, for each i, $f^{\square V(T_i)}$ is a (trivial) cofibration in $\mathcal{V}^{G\times\mathrm{Aut}(T_i)}_{\mathcal{F}_{T_i}}$ (where we are using that $\mathrm{Aut}(T_i) \subseteq \mathrm{Aut}(T)$, and hence $f|_{V(T_i)}$ satisfies the necessary assumptions for T_i).

- By Proposition 5.3.25, we have

$$\left(f^{\square V(T_i)} \right)^{\square k_i} \in \mathcal{V}^{G\times\Sigma_{k_i}\wr \mathrm{Aut}(T_i)}_{\mathcal{F}^{\otimes_G k_i}_{T_i}}$$

  is a (trivial) cofibration, for each i.

- By Lemma 5.3.23, we have

$$\mathop{\square}_{i=1}^{r}\left(\left(f^{\square V(T_i)}\right)^{\square k_i}\right) \in \mathcal{V}^{G\times\prod\Sigma_{k_i}\wr\mathrm{Aut}(T_i)}_{\mathcal{F}_{T_1}^{\otimes_G k_1}\sqcap_G\cdots\sqcap_G\mathcal{F}_{T_r}^{\otimes_G k_r}}$$

  is a (trivial) cofibration.

- Finally, after applying Lemma 5.3.8 to show that $f(v_r)$ is a (trivial) cofibration in $\mathcal{V}^{G\times\mathrm{Aut}(T)}_{\pi_r^*(\mathcal{F}_n)}$, and using Lemmas 2.3.36 and 5.3.26 identifying the target category below, by Lemma 5.3.11 we have

$$f^{\square V(T)} = f(v_r)\square\mathop{\square}_{i=1}^{r}\left(\left(f^{\square V(T_i)}\right)^{\square k_i}\right) \in \mathcal{V}^{G\times\mathrm{Aut}(T)}_{\mathcal{F}_T}$$

  is a (trivial) cofibration, as desired.

$\square$

Remark 5.3.28. As stated earlier, we expect to be able to use the filtration and analysis above to build the $\mathcal{F}$-model structure on $\mathcal{V}\mathsf{Op}^G_{\mathfrak{C}}$ in certain interesting cases, such as $\mathcal{V} = \mathsf{sSet}$, Top, or Ch_R. As is usually the case, this will require relaxing the dependency on $\mathcal{P}$ being $\mathcal{F}$-cofibrant.

Chapter 6

Genuine Equivariant Operads

Again, we fix a finite group G, as well as an indexing system $\mathcal{F} = \{\mathcal{F}_n\}$, which will be weak unless otherwise stated.

In this chapter, we introduce a new algebraic framework called "genuine $\mathcal{F}$-equivariant operads". These objects encode operations with specific isotropy, and record precisely how the equivariance of components determines the equivariance of their composition.

We begin by motivating the need and desire for such structures. We will then define multiple categorical models for this structure, as well as a preliminary list of examples, including the homotopy genuine G-operad associated to a G-∞-operad.

Research in this topic is ongoing; in particular, complete descriptions and analysis of algebras over these structures, as well as a complete comparison of the models below, are in progress.

6.1 Motivation

There are three main motivations for the necessity and construction of genuine equivariant operads:

First, recall our discussion at the end of Section 4.2.3. One of the downfalls of that analysis was that the operadic information for different subgroups $H \leq G$ did not interact with each other, even though we know that the composition structure maps for G-operads have an additional level of equivariance which pulls from different subgroups (see Diagram

(4.1)). However, this information cannot be recorded in the structure of a G-operad, as we have competition between the number of inputs allowed verses the number of *orbits of inputs* desired, as specified by the equivariance of the root operation. We desire a notion of operad with structure maps that record the initial equivariance, and restrict the possible inputs to only have that orbital structure.

Second, recall that the natural diagram connecting G-spaces and coefficient systems of sets

$$\begin{tikzcd} \mathsf{Set}^G \arrow[d] & \mathsf{Top}^G \arrow[l, "\pi_0"'] \arrow[d, "\Phi"] \arrow[dl, "(\pi_0)_G"] \\ \mathsf{Set}^{O_G^{op}} & \mathsf{Top}^{O_G^{op}} \arrow[l, "\pi_0"] \end{tikzcd}$$

does *not* commute. In particular, the G-set can miss important homotopical fixed-point information, which is not discarded by the coefficient system of sets.

Analogously, consider the top row of the diagram below.

$$\begin{tikzcd} \mathsf{Op}^G \arrow[d] & \mathsf{sOp}^G \arrow[l, "\pi_0"'] \arrow[d, "\Phi'"] \arrow[dl, "(\pi_0)_G"] \\ \mathcal{C}_1 & \mathcal{C}_2 \arrow[l, "\pi_0"] \end{tikzcd}$$

Again, $\pi_0 : \mathsf{sOp}^G \to \mathsf{Op}^G$ misses homotopical fixed-point information, particularly that $\pi_0(\mathcal{O}(n)^\Gamma) \not\simeq (\pi_0\mathcal{O}(n))^\Gamma$. As in the diagram above, there should be two categories, denoted $\mathcal{C}_1$ and $\mathcal{C}_2$ in the diagram above, which play the role of $\mathsf{Set}^{O_G^{op}}$ and $\mathsf{Top}^{O_G^{op}}$ — taking into account the combinatorial complexities of operads, of course. Note that the naive guess of $\mathsf{Op}^{O_G^{op}}$ does *not* suffice: it allows us to record the homotopical information of $\mathcal{O}(n)^H$, but not of any of the non-trivial graph subgroups $\mathcal{O}(n)^\Gamma$. Incorporating control over those subgroups while maintaining an operadic structure will require a new organizational framework.

Similarly, when we built the homotopy operad associated to an inner G-Kan complex in Section 5.2.4, we were only able to record the information about G-free compositions, and all others were discarded. We desire a category of operads which allows for strictification of G-∞-operads which incorporates all of the information present.

Third, after discovering the category of (planar) G-trees, it became clear that they had an algebraic structure analogous to that for regular planar trees. However, just as with the homotopy G-operad above, no amount of manipulation or categorical gymnastics would allow it to have the structure of a coloured operad. The grafting structure of G-trees both inspired and motivation our definitions below, as well as provided visual heuristics for understanding.

As with operads, the new structure comes in many equivalent descriptions, generalizing different ideas we have seen throughout this thesis. We devote this chapter to identifying and comparing these models.

Remark 6.1.1. Two words of warning.

(1) Unlike the non-operadic cases of equivariant spaces, categories, or spectra, the category of "genuine equivariant $\mathcal{V}$-operads" will not be a category of (enriched) presheaves on $\mathcal{V}\mathsf{Op}$. Instead, the comparison between $\mathcal{V}\mathsf{Op}^G$ and the genuine operads will happen on the underlying symmetric sequences.

(2) Every model currently understood has the additional requirement that the underlying symmetric monoidal category $\mathcal{V}$ is in fact *Cartesian*. We hope to remove this barrier in the future.

Remark 6.1.2. The different definitions below serve different utilities. Some of them are more straightforward to state, while others require significant technology. Some are immediately able to capture both single- and multi-coloured operads, as well as restrictions to any weak indexing system $\mathcal{F}$, while others will require significant effort for either generalization. Some are monoids, some are algebras, and some are neither. Some we expect to be easier to endow with model structures, while others will be more troublesome. For completeness, we present them all.

6.1.1 Planar G-Trees and G-Symmetric Sequences

Two models (the "monadic" and "coend") are generalizations of the idea that operads are "symmetric sequences with structure", whether that structure is as a monoid over the composition product, or an algebra over the free operad monad. To begin, we define the equivariant version of this underlying structure. This requires us to first discuss planar G-trees.

We remark that the definition of a planar structure on a tree $T \in \Omega$ from Section 2.3.4 extends immediately to the category of forests, by providing a planar structure on each component and a total ordering of the root orbit.

Definition 6.1.3. A *planar* G-tree is a G-tree T along with a planarization of the underlying forest. Let $\mathbb{T}_G$ denote a choice of category of planar G-trees and non-planar morphisms, such that there is exactly one representative of each planarization (c.f. Definition 2.3.42); both $\mathbb{T}_G^q$ and $\mathbb{T}_{G,0}$ will denote the wide subcategory of (non-planar) isomorphisms and quotients, and $\Sigma_G \subseteq \mathbb{T}_{G,0}$ will denote the full subcategory of G-corollas and (non-planar) quotients.

More generally, let $\mathbb{T}_\mathcal{F}$ denote the full subcategory of $\mathbb{T}_G$ spanned by $\mathcal{F}$-admissible trees, and $\Sigma_\mathcal{F}$ similarly.

Definition 6.1.4. Let $\mathbb{T}_{\mathcal{F},0}$ (resp. $\mathbb{T}_\mathcal{F}^t$) denote the wide subcategory of planar G-trees with G-isomorphisms (respectively *planar tall* G-maps).

Remark 6.1.5. We saw for non-equivariant trees that the condition $\varphi(L(S)) = L(T)$ and $\varphi : L(S) \to L(T)$ is a bijection are equivalent. However, this is *not* the case for G-trees. In particular, in addition to inner faces and degeneracies, quotient maps are considered tall.

To clarify, we say a map is (planar) *strictly tall* if it is planar tall and $\varphi : L(S) \to L(T)$ is a bijection; denote this wide subcategory by $\mathbb{T}_G^{st}$.

Lemma 6.1.6 (c.f Lemma 5.1.45)**.** *The following categories are equivalent (and all have some notion of "planarity"):*

(1) the category of G-corollas Σ_G;

(2) the Grothendieck construction on the functor

$$\begin{aligned} O_G^{op} &\longrightarrow \mathsf{Cat} \\ G/H &\longmapsto \Sigma^{B_{G/H}G} \end{aligned}$$

(3) The disjoint union

$$\coprod_{n \geq 0} O_{\Gamma_n}$$

where $O_{\Gamma_n} \subseteq O_{G\times\Sigma_n}$ is the subcategory generated by the family of graph subgroups; i.e. spanned by the $G \times \Sigma_n/\Gamma$ for Γ a graph subgroup.

□

Lemma 6.1.7. *The following categories are equivalent:*

(1) the category of $\mathcal{F}$-corollas Σ_F;

(2) The disjoint union

$$\coprod_{n \geq 0} O_{\mathcal{F}_n}$$

where $O_{\mathcal{F}_n} \subseteq O_{G\times\Sigma_n}$ is the subcategory spanned by the $G \times \Sigma_n/\Gamma$ for Γ a graph subgroup in $\mathcal{F}_n$.

□

As before, the functor which forgets the planar structure yields an equivalence of categories $\Sigma_G \simeq \Upsilon_G$, so in fact Σ_G is also equivalent to all the categories listed in Lemma 5.1.45.

Our valence functor $val : \Omega_G^q \to \Upsilon_G$ extends naturally to the planar categories $val : \mathbb{T}_{G,0} \to \Sigma_G$. Similarly, if $T \in \mathbb{T}_G$ and $G.v \in V_G(T)$, we (abusively) denote by $T_{G.e}$ the G-corolla with its inherited planar structure.

Remark 6.1.8. As in the non-equivariant case, val is not a Grothendieck construction; if the data of a object included an additional independent total ordering on the leaves, it would be.

Remark 6.1.9. The inclusion of G-corollas is again a section $s : \Sigma_G \to \mathbb{T}_G$ of val, extending the non-planar section $s : \Upsilon_G \hookrightarrow \Omega_G$.

Lemma 6.1.10. *Strictly planar tall maps, if they exist, are unique between two G-trees. Further, if $val(T) = C$ for some G-corolla C, then there is a (necessarily unique) planar strictly tall map $C \to T$.* □

Now, recall that Sym^G denoted the category $\mathrm{Fun}(\Sigma \times G, \mathcal{V})$ of symmetric G-sequences. We generalize these in a natural way.

Definition 6.1.11. A *G-symmetric sequence* in $\mathcal{V}$ is a functor $X : \Sigma_G^{op} \to \mathcal{V}$, and we denote the category of such functors by Sym_G.

The natural inclusion $i : \mathbb{T} \times G \to \mathbb{T}_G$ restricts to one $i : \Sigma \times G \hookrightarrow \Sigma_G$, yielding a diagram

$$\begin{tikzcd} \mathsf{Sym}^G \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \mathsf{Sym}_G \arrow[l, "i^*"'] \end{tikzcd}$$

Notation 6.1.12. If X is a G-symmetric sequence, following Lemma 5.1.46 we may denote $X(C_A)$ by $X(A)$, for some (totally ordered) $A \in \mathsf{F}^H$, $H \leq G$.

Remark 6.1.13. We observe that the target category of G-symmetric sequences is $\mathcal{V}$, as opposed to $\mathcal{V}^G$. This is because Σ_G is acting as both Σ *and* G when compared against symmetric G-sequences $\mathrm{Fun}(\Sigma \times G, \mathcal{V}) = \mathrm{Fun}(\Sigma, \mathcal{V}^G)$.

6.2 Sheaf Model

We begin with the model which is by far the easiest to state, is simple to motivate, and is defined in the largest of generalities.

Definition 6.2.1. A *(sheaf) genuine G-operad* is a genuine equivariant dendroidal set $X \in \mathsf{dSet}_G$ which satisfies a strict Segal condition:

$$X(T) \simeq \lim_{\mathsf{Out}_c^G(T)} X = \prod_{v \in V_G(T)} X(T_{G.v})/\sim := \lim_{v \in V(T)} X(T_v) \tag{6.1}$$

where $\sim$ is induced by edges.

In the special case where $X(G \cdot_H \eta) = *$ for all H, this says that, for all T, $X(T)$ is *isomorphic* to the product of its evaluation on the vertices of T.

More generally, a *(sheaf) genuine G-operad* in $\mathcal{V}$ is a presheaf $X \in \mathcal{V}^{\Omega_G^{op}}$ such that $X(T)$ is *isomorphic* to the product of its evaluation on the vertices of T (modulo edge relations).

Most generally, a *(sheaf) genuine $\mathcal{F}$-operad* in $\mathcal{V}$ is a presheaf $X \in \mathcal{V}^{\Omega_{\mathcal{F}}^{op}}$ such that $X(T)$ is *isomorphic* to the product of its evaluation on the vertices of T (modulo edge relations).

We say a sheaf genuine G-operad is *single-coloured* if $X(G \cdot_H \eta) = *$ for all $H \leq G$; otherwise, X is called *multicoloured*.

Remark 6.2.2. In the non-equivariant case, the analogous condition to (6.1) holds for $X \in \mathsf{dSet}$ if and only if X is the nerve of an operad (Lemma 2.4.8). However, this need not be true in the equivariant case, for the reasons outlined above, and exemplified in the simple Example 6.3.29. Moreover, we also note that this definition manifestly requires $\mathcal{V}$ to be Cartesian.

Algebraically, given any sheaf genuine G-operad X, we have the following necessary data:

(1) a *coefficient system* of colours $\mathfrak{C}_X : O_G^{op} \to Set$ given by $G/H \mapsto X(G/H \cdot \eta)$;

(2) colour-coordinated multiplicative structure maps, induced by inner faces;

$$X(T) \simeq \lim_{v \in V(T)} X(T_v) \to X(val(T))$$

(3) a unit per colour, induced by the degeneracy maps.

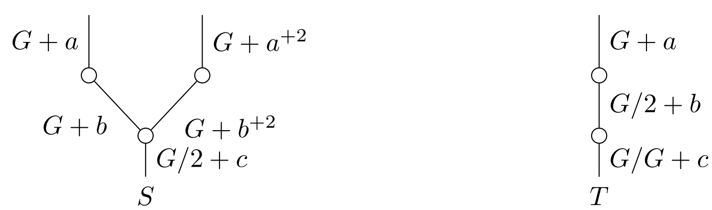
$$X(G/H \cdot \eta) \to X(C_{H/H})$$

This structure is subject to the conditions that the multiplication maps are associative (via composition schema for inner faces), unital (via composition schema for degeneracies), T-equivariant (via actions by $\mathrm{Aut}(T)$), and natural with respect to quotient maps.

Remark 6.2.3. We note that this is an obvious strictification of G-∞-operads (cf. 5.2.18), which requires composition to be truly well-defined, but has relaxed the equivariant restrictions. Specifically, we have not enforced any strict fixed-point conditions: each $X(G \cdot T/(-))$ is a proper coefficient system in $O_{\Gamma(G,\mathrm{Aut}(T))}$.

We briefly unpack this last condition of "quotient naturality" in the special case that our genuine operad is single-coloured.

Example 6.2.4. Let $G = \mathbb{Z}/4$, and consider the following quotient map $q : S \to T$.



(6.2)

Any genuine $\mathbb{Z}/4$-operad would have structure maps γ below such that the diagram commutes:

$$\begin{array}{ccc} N_X(T) = X(G/2G) \times X(2G/e) & \xrightarrow{\gamma} & X(G/e) \\ \downarrow q^* & & \downarrow q^* \\ N_X(S) = X(2G/2G \amalg 2G/2G) \times X(2G/e) \times X(2G/e) & \xrightarrow{\gamma} & X(2G/e \amalg 2G/e) \end{array}$$

where q^* is the diagonal on vertices which combine under the quotient map, and we have denoted $X(A)$ for $X(C_A)$ and any H-set A.

We importantly observe that this quotient naturality extends over operations with different "numbers of inputs". This follows from the fact that, even though the resulting operation has the same "cardinality" of four, what is most important to count is the number of *orbits* in target H-set. The fact that this number is not preserved under restriction is what has required us to define these algebraic objects: recording the equivariance of the operations properly requires us to step away from rigidly counting inputs.

Example 6.2.5. Given any operad $\mathcal{O} \in \mathcal{V}\mathsf{Op}^G$, the push-forward of the nerve $i_* N\mathcal{O} \in \mathsf{dSet}_G$ is a genuine G-operad.

We will explore the more algebraic comparison between operads and genuine operads below in Section 6.3.4.

We observe that this definition is quite robust:

(1) it is efficiently defined;

(2) it encodes both single-coloured and multicoloured genuine G-operads;

(3) it is well-defined for any weak indexing system $\mathcal{F}$.

However, while this definition is easy to state and quite well motivated, it is difficult to prove theorems about, and can be difficult to work with on its own. For example, it does not immediately lend itself to a nice definition of algebras, nor is it clear what the correct model structure of genuine G-operads should be: for example colimits are not constructed underlyingly in dSet, indicating that our intuition from dSet is most likely incorrect.

As we move forward, we will keep this definition in mind.

6.3 Monad Model

Our next model of genuine G-operads will be constructed by generalizing technology we developed in Chapter 3.

Recall from Section 3.2 that single-coloured $\mathcal{V}$-operads are equivalent to $\mathbb{F}$-algebras, where $\mathbb{F}$ is the monad on $\mathcal{V}^{\Sigma^{op}}$ defined by the left Kan extension of N_X over val. This story generalizes to the equivariant setting.

Warning 6.3.1. As mentioned above, we require our category $\mathcal{V}$ to be *Cartesian monoidal*. In this model, the quotient maps in Ω_G will require our category $\mathcal{V}$ to have functorial arrows of the form

$$(\mathsf{F} \wr \mathcal{C}^{op})^{op} \to \mathcal{C},$$

exploiting diagonal maps.

Remark 6.3.2. The entire discussion below also works when restricted to $\mathbb{T}_{\mathcal{F},0}$ and $\Sigma_{\mathcal{F}}$ for any weak indexing system $\mathcal{F}$ (using that, by definition, *val* restricts to a functor between these two categories). For ease of notational burden and convenience, we will work with the strongest possible structure of genuine G-operads (i.e., with $\mathcal{F} = \underline{\mathsf{Set}}$).

This should also work (with minor adjustments, cf. 4.2.4) in the G-coloured context, though that will require further meticulous, if not challenging, elaborations of the technology below.

We observe that planar G-trees provide salient examples of elements of functors to the wreath category $\mathsf{F} \wr \Sigma_G$, where we recall F is the category of finite sets.

Definition 6.3.3. Given $T \in \mathbb{T}_G$, define the vertex functor $\mathbb{V} : \mathbb{T}_{G,0} \to \mathsf{F} \wr \Sigma_G$ by $T \mapsto (V_G(T), val)$ where val sends $G.v$ to the G-corolla $T_{G.v}$.

Note that by including quotient maps, we can no longer land in $\mathsf{F} \wr \Sigma_G$, as these maps are surjections (and certainly not always injections).

Definition 6.3.4. Given a G-symmetric sequence Y, the *nerve evaluation* functor, denoted

$$N_Y : \mathbb{T}_{G,0}^{op} \to \mathcal{V},$$

is defined as the composite

$$\mathbb{T}_{G,0}^{op} \xrightarrow{\mathbb{V}} (\mathsf{F} \wr \Sigma_G)^{op} \xrightarrow{\mathsf{F} \wr Y} (\mathsf{F} \wr \mathcal{V}^{op})^{op} \xrightarrow{\times} \mathcal{V}$$

Definition 6.3.5 (c.f. Definition 3.2.3)**.** Define $\mathbb{F}_G$ to be the endofunctor on G-symmetric sequences $X \in \mathcal{V}^{\Sigma_G^{op}}$ defined by the left Kan extension below.

$$\begin{tikzcd}
\mathbb{T}_{G,0}^{op} \arrow[r, "N_X"] \arrow[d, "val"'] \arrow[dr, phantom, "\Searrow"] & \mathcal{V} \\
\Sigma_G^{op} \arrow[ur, "\mathbb{F}_G X := \mathrm{Lan}_{val} N_X"'] &
\end{tikzcd}$$

Given a morphism $\varphi : X \to Y$, define $\mathbb{F}_G(\varphi)$ to be the map induced by Lemma A.1.2 applied to the natural transformation below:

$$\begin{tikzcd}
\mathbb{T}_{G,0} \arrow[r, "\mathbb{V}"] \arrow[d, "val"'] & \mathsf{F} \wr \Sigma \arrow[r, bend left, "\mathsf{F}\wr X"] \arrow[r, bend right, "\mathsf{F}\wr Y"'] \arrow[r, phantom, "\Downarrow \mathsf{F}\wr\varphi"] & \mathsf{F} \wr \mathcal{V} \arrow[r] & \mathcal{V} \\
\Sigma \arrow[urrr, dotted, bend right] \arrow[urrr, dotted, bend right=40] & & &
\end{tikzcd}$$

The main result of this section is the following:

Theorem 6.3.6. *$\mathbb{F}_G$ has the structure of a monad.*

Definition 6.3.7. We call algebras over $\mathbb{F}_G$ *(monadic) genuine equivariant operads.*

We will define the structure maps of this monad, generalizing what we observed in Section 3.2

We begin by lifting the definitions in Section 3.1.1 to the setting of G-trees.

Definition 6.3.8. Let $\mathbb{T}_{G,1}$ denote the pullback in categories

$$\begin{tikzcd}
\mathbb{T}_{G,1} \arrow[r, "\mathbb{V}"] \arrow[d, "d_1"'] & \mathsf{F} \wr \mathbb{T}_{G,0} \arrow[d, "\mathsf{F}\wr val"] \\
\mathbb{T}_{G,0} \arrow[r, "\mathbb{V}"] & \mathsf{F} \wr \Sigma_G
\end{tikzcd}$$

Explicitly, objects in $\mathbb{T}_{G,1}$ are G-trees with equivariant assembly data: each vertex orbit $G.v$ is equipped with a G-tree $S^{G.v}$ such that $val(S^{G.v}) = T_{G.v}$. Equivalently, we have a collection of planar tall maps $T_{G.v} \to S^{G.v}$. As before, this data can equivalently be packaged as a functor

$$S^{(-)} : \mathrm{Out}_c^G(T) \to \mathbb{T}_{G,0}$$

and a natural transformation $U : id \Rightarrow S^{(-)}$ of planar tall G-maps.

We additionally have assembly maps $d_0 : \mathbb{T}_{G,1} \to \mathbb{T}_{G,0}$. Indeed, we note that the proof of Proposition 3.1.12 generalizes immediately to handle forests, and since the assembly data in $\mathbb{T}_{G,1}$ are equivariant, the colimit will again be a G-tree:

Proposition 6.3.9. *Given $(T, (S^{G.v})) \in \mathbb{T}_{G,1}$, the colimit*

$$T \wedge (S^{G.v}) := \operatorname*{colim}_{V_G(T)} S^{(-)}$$

in the category $\mathbb{T}_{G,0}$ exists. □

We have two sections s_0 and s_{-1} of d_0:

$$s_0(T) = (T, (T_{G.v})) \text{ is the trivial assembly data, and}$$

$$s_{-1}(T) = (val(T), T) \text{ is the co-trivial assembly data.}$$

We note that s_{-1} is *not* a unit of d_1, but instead we have a commuting diagram

$$\begin{tikzcd}
\mathbb{T}_{G,0} \arrow[r, "val"] \arrow[d, "s_{-1}"'] & \Sigma_G \arrow[d, "s"] \\
\mathbb{T}_{G,1} \arrow[r, "d_1"] & \mathbb{T}_{G,0}
\end{tikzcd}$$

As in Lemma 3.2.6, we can build iterates of this endofunctor by using these pullbacks.

Lemma 6.3.10 (c.f. Lemma 3.2.6)**.** *Given any G-symmetric sequence X, the sequence $\mathbb{F}_G\mathbb{F}_G X$ is isomorphic to the left Kan extension*

$$\mathbb{F}_G\mathbb{F}_G X \simeq \mathrm{Lan}_{val \circ d_1}(\times \circ \mathsf{F} \wr N_X \circ \mathbb{V}).$$

Proof. The proof follows exactly as it does for Lemma 3.2.6. $\square$

We can moreover iterate this "assembly data" construction, as we did in Section 3.2.

Definition 6.3.11. Suppose we have define $\mathbb{T}_{G,k}$ for all $k < n$. Define $\mathbb{T}_{G,n}$ to be the pullback

$$\begin{tikzcd}
\mathbb{T}_{G,n} \arrow[r, "\mathbb{V}"] \arrow[d, "d_n"'] & \mathsf{F} \wr \mathbb{T}_{G,n-1} \arrow[d, "d_{n-1}"] \\
\mathbb{T}_{G,n-1} \arrow[r, "\mathbb{V}"] & \mathsf{F} \wr \mathbb{T}_{G,n-2}
\end{tikzcd}$$

We again have maps $d_0 : \mathbb{T}_{G,n} \to \mathbb{T}_{G,n-1}$, created by assembling the last two components together. Moreover, the proof of Lemma 3.1.16 immediately generalizes to the equivariant setting:

Lemma 6.3.12. *$\mathbb{T}_{G,n}$ is equivalent to the category of n-fold strings of planar tall G-maps, with quotients connecting strings. Moreover, these $\mathbb{T}_{G,*}$ form a simplicial object in categories, and the assembly maps are induced by the simplicial morphisms.* $\square$

Now, let us construct the monad structure maps.

Definition 6.3.13. Given any G-symmetric sequence X, define the monad multiplication natural transformation $\gamma_X : \mathbb{F}_G\mathbb{F}_G X \Rightarrow \mathbb{F}_G X$ to be the arrow induced by Lemma A.0.2 from the (opposite of the) natural transformation in $\mathsf{WSpan}(\Sigma_G, \mathcal{V})$ below:

$$\begin{tikzcd}
\mathbb{T}_{G,1} \arrow[r, "\mathbb{V}"] \arrow[d, "\pi_1"'] \arrow[drr, Rightarrow, "\Phi_p"] & \mathsf{F} \wr \mathbb{T}_{G,0} \arrow[r, "\mathsf{F}\wr\mathbb{V}"] & \mathsf{F}^{\wr 2}\Sigma_G \arrow[r, "\mathsf{F}^{\wr 2}X"] \arrow[d, "\mathrm{coll}"] & \mathsf{F}^{\wr 2}\mathcal{V}^{op} \arrow[r, "\mathsf{F}\wr\otimes"] \arrow[d, "\mathrm{coll}"] \arrow[drr, Rightarrow, "\Phi_\times"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\times"] & \mathcal{V}^{op} \arrow[d, equal] \\
\mathbb{T}_{G,0} \arrow[rr, "\mathbb{V}"'] \arrow[d, "val"'] & & \mathsf{F} \wr \Sigma_G \arrow[r, "\mathsf{F}\wr X"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr, "\times"'] & & \mathcal{V}^{op} \\
\Sigma_G & & & & &
\end{tikzcd} \tag{6.3}$$

The monad unit $\epsilon_X : X \to \mathbb{F}_G X$ is defined by the arrow induced by Lemma A.0.2 from the (opposite of the) identity natural transformation below, where we recall σ^i from the discussion after 3.1.2.

$$
\begin{tikzcd}
\Sigma_G \arrow[rrrr, "X"] \arrow[d, "s"] \arrow[dr, "\sigma^0"] & & & & \mathcal{V}^{op} \arrow[d, equal] \\
\mathbb{T}_{G,0} \arrow[r, "\mathbb{V}"] \arrow[d, "val"] & \mathsf{F}\wr\Sigma_G \arrow[r, "\mathsf{F}\wr X"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr, "\times"] & & \mathcal{V}^{op} \\
\Sigma_G & & & &
\end{tikzcd}
\tag{6.4}
$$

proof of Theorem 6.3.6. We begin by showing unitality. To start, we note that the maps $\mathbb{F}_G\epsilon_X$ and $\epsilon_{\mathbb{F}X}$ are given by the (opposite of the) top face and back face, respectively, of the following diagrams, induced by Lemma A.1.2.

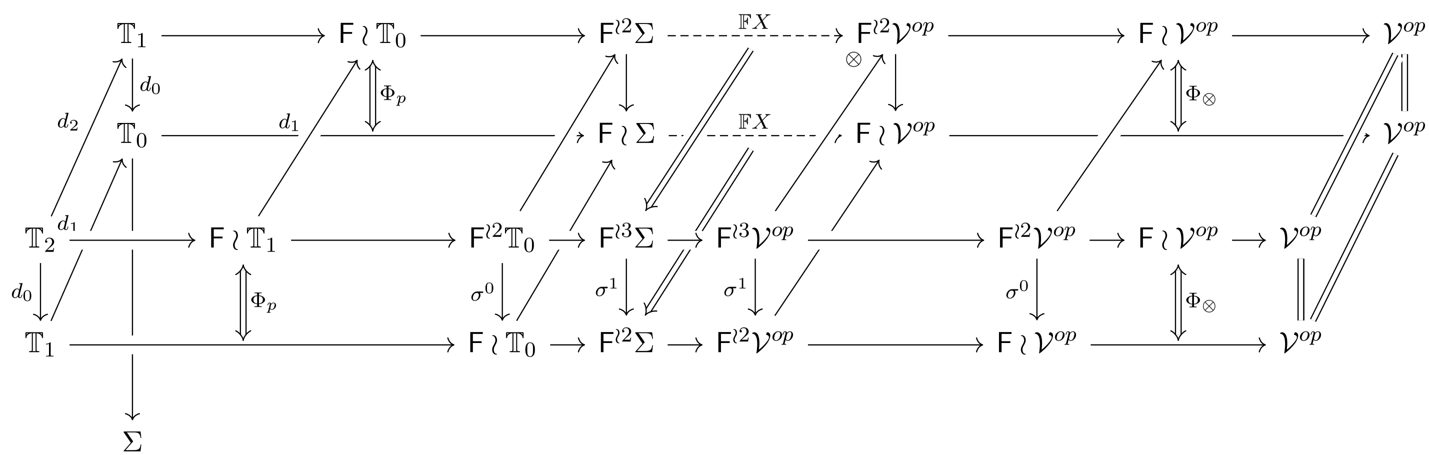
$$
\begin{tikzcd}
& & \Sigma_G & & & & & & \\
& \mathbb{T}_{G,0} \arrow[ur, "val"] \arrow[rr, "\mathbb{V}"] & & \mathsf{F}\wr\Sigma_G \arrow[rrr, "\mathbb{F}_G X"] & & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr, "\times"] & & \mathcal{V}^{op} \\
\mathbb{T}_{G,0} \arrow[ur, equal] \arrow[rr, "\mathbb{V}"] \arrow[d, "s_0"'] & & \mathsf{F}\wr\Sigma_G \arrow[ur, equal] \arrow[rrr, "X"] \arrow[d, "s"'] \arrow[drr, "\sigma^0"] & & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[ur, equal, "\times"'] \arrow[rr, "\times"] \arrow[d, "\sigma^0"'] \arrow[drr, equal] & & \mathcal{V}^{op} \arrow[ur, equal] & \\
\mathbb{T}_{G,1} \arrow[uur, dashed, "d_1"'] \arrow[rr, "\mathbb{V}"] & & \mathsf{F}\wr\mathbb{T}_{G,0} \arrow[uur, dashed, "val"'] \arrow[rr, "\mathbb{V}"] & & \mathsf{F}^{\wr 2}\Sigma_G \arrow[r, "X"] & \mathsf{F}^{\wr 2}\mathcal{V}^{op} \arrow[uur, bend right] \arrow[rr, "\times"] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[u, "\times"'] \arrow[uur, bend right, "\times"'] &
\end{tikzcd}
\tag{6.5}
$$

$$
\begin{tikzcd}
& \Sigma_G \arrow[rrrrrrr, dashed, "\mathbb{F}X"] \arrow[dd] & & & & & & & \mathcal{V}^{op} \arrow[dl, equal] \arrow[dd, equal] \\
\mathbb{T}_{G,0} \arrow[ur, "val"] \arrow[rr] \arrow[dd, "s_{-1}"'] \arrow[ddrr, bend right, "\sigma^0"] & & \mathsf{F}\wr\Sigma \arrow[rrr] \arrow[ddrr, bend left, "\sigma^1"] & & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] \arrow[dd, "\sigma^1"'] \arrow[ddr, equal] & & \mathcal{V}^{op} \arrow[dd, equal] & \\
& \mathbb{T}_{G,0} \arrow[rr, no head] \arrow[dd] & & \mathsf{F}\wr\Sigma \arrow[rrrrr, dotted, "\mathsf{F}\wr\mathbb{F}_G X"] & & & & & \mathcal{V}^{op} \arrow[dl, equal] \\
\mathbb{T}_{G,1} \arrow[ur, "d_1"] \arrow[rr] & & \mathsf{F}\wr\mathbb{T}_{G,0} \arrow[ur, "val"] \arrow[rr] & & \mathsf{F}^{\wr 2}\Sigma \arrow[r, "X"] & \mathsf{F}^{\wr 2}\mathcal{V}^{op} \arrow[r] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} & \\
& \Sigma_G & & & & & & &
\end{tikzcd}
\tag{6.6}
$$

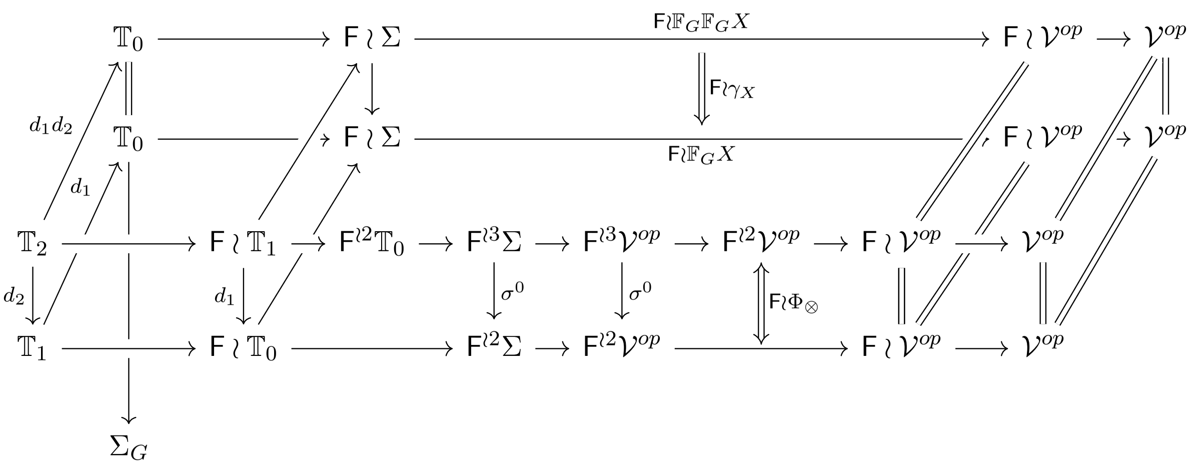
It is a straightforward verification that both of these diagrams (of natural transformations) commute. More, we observe that the curved bottom face in Diagram 6.5 is a left Kan extension, by combining Lemmas A.0.2, A.1.8, and A.1.9; similarly, both the top and bottom faces of 6.6 are left Kan extensions. Thus, by Lemma A.1.6, the desired maps are induced by the left Kan extension over the front faces.

Hence, to check unitality, it suffices to show that the composition of these front faces stacked on top of the natural transformation in Diagram (6.3) is the identity; this too is straightforward. Hence their left Kan extensions produce the same maps, proving unitality.

For associativity, we observe that the natural maps $\gamma_{\mathbb{F}_G X}$ and $\mathbb{F}_G(\gamma_X)$ are induced via left Kan extension by the back face of the following diagrams.



(6.7)



(6.8)

In both Diagrams (6.7) and (6.8), the top and bottom faces are left Kan extensions, via Lemmas A.0.2, A.1.8, and A.1.9. Moreover, the left-most box in (6.7) with front and back sides both filled by the 2-cell Φ_p commutes, as the re-planarization is ignoring the last piece of assembly data. Similarly, we verify that these entire diagrams of natural transformations commute. Thus, again by Lemma A.1.6, we have that the desired maps are induced by the front faces.

Finally, it is a straightforward verification that the horizontal composites of these faces with the natural transformation from Diagram (6.3) are equal. Hence, their left Kan extensions produce the same maps, and thus our monad is associative, as desired. □

Definition 6.3.14. A *genuine equivariant operad* is an $\mathbb{F}_G$-algebra in the category of G-symmetric sequence. We denote the category of $\mathbb{F}_G$-algebras by Op_G.

The structure of being a genuine equivariant operad can be unpacked in many ways. First, we have the following:

Lemma 6.3.15 (cf. Lemma 3.2.9)**.** *An $\mathbb{F}_G$-algebra structure on X is equivalent to the data of a morphism $\tilde{\mu} : N_X \Rightarrow X \circ val$ such that*

(1) (unitality) $\tilde{\mu}$ is the identity on all G-corollas; and

(2) (associativity) the following two (compositions of) natural transformations are equal:

$$\begin{tikzcd}
\mathbb{T}^{op}_{G,1} \arrow[r, "\mathbb{V}"] \arrow[d, "d_1"'] & \mathsf{F} \wr \mathbb{T}^{op}_{G,0} \arrow[r, "\mathsf{F}\wr N_X"] \arrow[d, "val"'] \arrow[dr, Rightarrow, "\mathsf{F}\wr\tilde{\mu}"] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\times"] \arrow[d, equal] & \mathcal{V} \arrow[d, equal] \\
\mathbb{T}^{op}_{G,0} \arrow[r, "\mathbb{V}"] \arrow[d, "val"'] & \mathsf{F} \wr \Sigma^{op}_G \arrow[r, "\mathsf{F}\wr X"] \arrow[d, phantom, "\Downarrow \tilde{\mu}"] & \mathsf{F} \wr \mathcal{V} \arrow[r] & \mathcal{V} \arrow[d, equal] \\
\Sigma^{op}_G \arrow[rrr, "X"] & & & \mathcal{V}
\end{tikzcd} \tag{6.9}$$

and

$$\begin{tikzcd}
\mathbb{T}^{op}_{G,1} \arrow[r, "\mathbb{V}"] \arrow[d, "d_0"'] & \mathsf{F} \wr \mathbb{T}^{op}_{G,0} \arrow[r, "\mathsf{F}\wr\mathbb{V}"] \arrow[rrr, bend left, "\mathsf{F}\wr N_X"] \arrow[d, phantom, "\Updownarrow \Phi_p"] & \mathsf{F} \wr \mathsf{F} \wr \Sigma^{op}_G \arrow[r, "\mathsf{F}\wr\mathsf{F}\wr X"] \arrow[d, "\mathrm{coll}"'] & \mathsf{F} \wr \mathsf{F} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\times"] \arrow[d, "\mathrm{coll}"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\times"] \arrow[d, phantom, "\Phi_\times \Updownarrow"] & \mathcal{V} \arrow[d, equal] \\
\mathbb{T}^{op}_{G,0} \arrow[rr, "\mathbb{V}"'] \arrow[d, "val"'] & & \mathsf{F} \wr \Sigma^{op}_G \arrow[r, "\mathsf{F}\wr X"'] \arrow[d, phantom, "\Downarrow \tilde{\mu}"] & \mathsf{F} \wr \mathcal{V} \arrow[rr, "\times"'] & & \mathcal{V} \arrow[d, equal] \\
\Sigma_G \arrow[rrrrr, "X"] & & & & & \mathcal{V}
\end{tikzcd} \tag{6.10}$$

where $\Phi_\times$ *and* Φ_p *are the re-collating and re-planarizing natural isomorphisms.*

□

More explicitly, a genuine equivariant operad is equipped with structure maps

$$\gamma : N_X(T) \to X(val(T))$$

for all G-trees T, which are associative (encoded by tall maps of G-trees), unital, T-equivariant, and natural under quotients. As is the case non-equivariantly (Remark 3.2.10), multiplicative unitality is encoded by a combination of the above conditions, observing the effect of assembly data containing sticks. Thus, such X have a whole *coefficient system* of units: for each stick $G/H \cdot \eta$, there is a map $\tilde{\mu} : * \to X(C_{H/H})$ (where $*$ is the empty product, i.e. the unit of our Cartesian $\mathcal{V}$).

Using these maps, and continuing to unpack definitions, we have the following results

Lemma 6.3.16. *Via unit, composition, and projection maps, the nerve evaluation* N_X *functor on genuine* G*-operads extends to all of* dSet*.* □

Proposition 6.3.17. *The functors* $X \mapsto N_X(-)$ *and* $Y \mapsto s^*Y$ *comparing* G*-symmetric sequences and genuine equivariant dendroidal sets restricts to an equivalence of categories between monadic genuine operads and sheaf genuine operads.* □

Lemma 6.3.18 (c.f. Lemma 3.1.10)**.** *For any genuine G-operad $\mathcal{O}$, $N_{\mathcal{O}}$ extends to a functor on $\mathbb{T}^t_G$, incorporating units and the structure maps. Moreover, $\mathcal{O}$ is isomorphic to the left Kan extension below.*

$$\begin{tikzcd}
\mathbb{T}_G^{t,op} \arrow[r, "N_{\mathcal{O}}"] \arrow[d, "val"'] \arrow[dr, phantom, "\Rrightarrow"] & \mathcal{V} \\
\Sigma_G^{op} \arrow[ur, "\mathrm{Lan}_{val}\, N_{\mathcal{O}} \cong \mathcal{O}"'] &
\end{tikzcd}$$

□

Remark 6.3.19. We could have generalized Definition 3.2.3 slightly differently, by instead replacing $\mathbb{T}_0$ with $\mathbb{T}_0^G$ (as opposed to $\mathbb{T}_{G,0}$). In this way, we would have defined $\mathbb{F}^G X$ to be the left Kan extension

$$\begin{tikzcd}
\mathbb{T}_0^G \arrow[r, Rrightarrow, "\mathbb{V}"] \arrow[d, "val"'] \arrow[dr, phantom, "\Rrightarrow"] & \mathsf{F}^G \wr \Sigma \arrow[r, "X"] & \mathsf{F}^G \wr \mathcal{V}^{op} \arrow[r, "\otimes"] & \mathcal{V} \\
\Sigma \arrow[urrr, "\mathbb{F}^G X"'] & & &
\end{tikzcd}$$

This would indeed capture some of the data we are looking for, in particular maps of the form in Equation (4.1), but just for the cases where $H = G$. So we would be able to analyze the equivariant information, but only in restricted settings. Indexing over all of $\mathbb{T}_{G,0}$ allows us to view all of this data simultaneously.

6.3.1 Modules and Algebras

These will be more naturally constructed in the "composition product" language of the upcoming section. However, we can also describe them in this context, using modifications of our machinery above.

Heuristically, modules over a genuine G-operad X will be sequences M with structure maps $(X, M)(T) \to M(val(T))$ for all G-trees T, while algebras will be modules "concentrated in degree 0".

Definition 6.3.20. We say a G-symmetric sequence Y is *concentrated in degree 0* if $Y = j_* Z$, for some $Z \in \mathcal{V}^{\mathcal{O}_G^{op}}$ and $j : \mathcal{O}_G \hookrightarrow \mathbb{T}_G$ is the inclusion of the equivariant 0-corollas.

Explicitly, $Y(C) = \varnothing$ unless the underlying G-forest of the corolla C has no leaves.

Definition 6.3.21. Let $\lambda^2\mathbb{T}_{G,0}$ denote the category of planar G-trees with a labeling of the vertices $\mathrm{Out}_c(T) \to \{a, m\}$, where "$a$" stands for "active" and "m" for "module", and isomorphisms which preserve these labelings.

Let $\lambda_1^1\mathbb{T}_G^t$ denote the category of labeled planar G-trees and maps which are the identity on m-labeled nodes, and tall on a-labeled nodes. Rigorously, for such a map $f : S \to T$, if the vertex $e^\uparrow \le e$ is labeled a (respectively m), then the subtree image $T_{f(e^\uparrow \le e)}$ (Definition 2.3.20) has all vertices labeled by a (respectively, is isomorphic to $S_{e^\uparrow \le e}$ and is labeled m).

Finally, let $\lambda_1^1\mathbb{T}_G^t(0)$ denote the full subcategory of $\lambda_1^1\mathbb{T}_G^t$ of those trees such that the underlying G-forest has no leaves.

We note that given a genuine G-operad X and a G-symmetric sequence M (respectively, Y concentrated in degree 0), we have a natural nerve evaluation maps

$$N_{(M),X} : \lambda_1^1\mathbb{T}_G^t \to \mathcal{V}$$
$$N_{(Y),X} : \lambda_1^1\mathbb{T}_G^t(0) \to \mathcal{V}.$$

Definition 6.3.22. Given a G-symmetric sequence M, an X*-module structure* on M is a natural transformation

$$\begin{tikzcd} \lambda_1^1\mathbb{T}_G^t \arrow[r, "N_{(M),X}"] \arrow[d, "val"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\ \Sigma_G^{op}. \arrow[ur, "M"'] & \end{tikzcd}$$

If M is concentrated in degree 0, then an X*-algebra structure* on M is a natural transformation

$$\begin{tikzcd} \lambda_1^1\mathbb{T}_G^t(0) \arrow[r, "N_{(M),X}"] \arrow[d, "val"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\ \Sigma_G(0)^{op} = O_G^{op}. \arrow[ur, "M"'] & \end{tikzcd}$$

Unpacking this somewhat, an X-algebra structure on M in particular encodes maps

$$X(C_{\amalg H/K_i}) \times \prod_i M(C_{K_i/\varnothing}) \to M(C_{H/\varnothing})$$

where $H/\varnothing$ is the empty H-set, which are unital, associative, $\Sigma-$ and G-equivariant, and natural under quotients.

Lemma 6.3.23. *If $G = e$, this recovers the usual notion of algebra over an operad.*

Proof. Restricting to trees of the form $C_m \circ (C_0)$ yields the usual form of the structure maps of an algebra M over an operad X; associativity and unitality are encoded by the naturality in the tall maps of $\lambda_1^1 \mathbb{T}_G^t(0)$. Conversely, collapsing all the edges connecting active nodes in a given tree in $\lambda_1^1 \mathbb{T}_G^t(0)$ yields precisely a tree of the form given above; it is clear that this induces the desired structure maps. □

The author expects to return to this structure very soon, and explore further what variety of structures can be encoded.

In particular, note the following:

Lemma 6.3.24. *Let $Z \in \mathsf{sSet}^G$ be thought of as a symmetric G-sequence in sSet concentrated in degree 0, and $\mathcal{F}$ a (strong) indexing system. Then the G-symmetric sequence $i_* Z$ (also concentrated in degree 0) is an $N_{\mathcal{F}}$-algebra if and only if Z is a strict $N^{\mathcal{F}}$-algebra, where $N_{\mathcal{F}}$ is the genuine $\mathcal{F}$-commutative operad and $N^{\mathcal{F}}$ is the N_∞-operad in simplicial sets associated to $\mathcal{F}$.*

Proof. We see that a $N^{\mathcal{F}}$-algebra structure on $i_* Z$ includes structure maps of the form

$$N_{\mathcal{F}}(C_{G/H}) \times A(C_{H/\varnothing}) \simeq A^{G/H} \to A$$

which are equivariant, associative, and unital. This is precisely the information of a strict $N^{\mathcal{F}}$-algebra. □

With that in mind, we expect the following to be true:

Conjecture 6.3.25. *Given a G-symmetric sequence M concentrated in degree 0 whose underlying G-symmetric sequence is cofibrant, and a (strong) indexing system $\mathcal{F}$, if M is an $N_{\mathcal{F}}$-algebra, then i^*M is an $N^{\mathcal{F}}$-algebra.*

This will be discussed further in Section 6.6

6.3.2 Examples of (Monadic) Genuine G-Operads

The classic examples of (G-)operads can be rebuilt in this context.

Example 6.3.26. The *commutative* genuine G-operad Comm is defined to be the constant G-symmetric sequence which evaluates to the unit of $\mathcal{V}$ on each level. Algebras over Comm are coefficient systems of commutative monoids in $\mathcal{V}$.

More generally, given any weak indexing system $\bar{\mathcal{F}} \subseteq \mathcal{F}$, define the *$\bar{\mathcal{F}}$-commutative* genuine G-operad $N_{\bar{\mathcal{F}}}$ to be the G-symmetric sequence which is the unit $\{*\}$ when evaluated on any $\bar{\mathcal{F}}$-admissible corollas, and empty otherwise.

Example 6.3.27. The *associative* genuine G-operad Assoc has underlying G-symmetric sequence $\mathcal{O}(C) = L(C) \simeq G \times \Sigma_n/N$. Similarly, algebras over Assoc are coefficient systems of associative monoids in $\mathcal{V}$.

Example 6.3.28. The *G-tree* genuine G-operad has underlying G-symmetric sequence $\mathcal{O}(C)$ given by all planar G-trees S with $val(S) = C$, with composition given by grafting, with units the sticks.

Example 6.3.29. If X is any coefficient system, consider the G-symmetric sequence which is equal to X on the 0-corollas, and is empty otherwise. It is immediate that this is a genuine operad. Algebras are just maps of coefficient systems out of X.

6.3.3 The Homotopy Genuine G-Operad

As the last motivating examples, we now build the homotopy genuine G-operad associated to any G-∞-operad $X \in \mathsf{dSet}^G$, where we restrict to the single-coloured case of $X(\eta, *) = \{*\}$.

This construction follows by carrying the results in [MW09] forward by using the following definitions, and liberally applying Lemma 5.2.29.

For any $X \in \mathsf{dSet}^G$, we recall that a dendrix $x \in X(T_0, *)$ is degenerate if the characterizing map factors $\Omega[T_0] \xrightarrow{\sigma} \Omega[S_0] \to X$ through a degeneracy map $\sigma : T_0 \to S_0$. For sheaves $Y \in \mathsf{dSet}_G$, we can make a similar definition:

Definition 6.3.30. Given $Y \in \mathsf{dSet}_G$ and $T \in \Omega_G$, a dendrix $y \in Y(T)$ is called G-*degenerate* if the characterizing map factors $\Omega[T] \xrightarrow{\sigma} \Omega[S] \to Y$ where $\sigma : T \to S$ is a degeneracy in Ω_G.

This is in fact an extension of the earlier definition, in the following manner.

Lemma 6.3.31. *Given $Y \in \mathsf{dSet}^G$, a G-tree T, and $y \in Y(T)$. Then a sub G-face of y is G-degenerate if and only if any associated orbital subface is degenerate.*

In particular, y itself is G-degenerate if and only if any associated orbital face is degenerate.

Proof. This follows by observing that the diagram below commutes, where $T \simeq G \cdot T_0/N$ and $S \simeq G \cdot S_0/M$.

$$\begin{tikzcd}
Y(G \cdot T_0) & Y(T) \arrow[l, "q^*"'] \\
Y(G \cdot S_0) \arrow[u, "(G\cdot\sigma')^*"] & Y(S) \arrow[l, "q^*"] \arrow[u, "\sigma^*"']
\end{tikzcd}$$

The top-right triangle commutes whenever ψ is the orbital face associated to the equivariant face φ. More, σ exists if and only if σ' exists, and when they do the lower-left triangle commutes. $\square$

This allows us to make the following definition unambiguously.

Definition 6.3.32. Let $X \in \mathsf{dSet}^G$ be a G-∞-operad. We define an equivalence relation on $i_*X(C)$ for all G-corollas C.

Let $A = \amalg H/K_i \amalg H/K \in \mathsf{Set}^H$, and $f, \bar{f} \in i_*X(C_A)$. We say that f is *homotopic* to $\bar{f}$ over H/K, written $f \sim_K \bar{f}$, if there exists $\gamma \in X(C_A \circ_K C_{K/K})$ such that

(1) the inner G-face $\partial_{G.e}\gamma \in i_*X(C_A)$ equals $\bar{f}$;

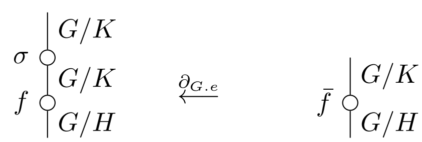
(2) every root cluster face $\partial_r\gamma \in i_*X(G \cdot_k C_1)$ is degenerate; and

(3) the leaf-cluster outer G-face $\partial_{G.v}\gamma \in i_*X(C_A)$ equals f,

where $G.e$ is the set of inner edges of $C_A \circ_K C_{K/K}$, and $G.v$ is the set of non-root vertices. We say γ *observes* the given homotopy.

Similarly, we say $f \sim_H \bar{f}$ if there exists $\gamma \in X(C_{H/H} \circ_H C_A)$ with analogous properties.

We can also record the properties of γ diagrammatically.



Remark 6.3.33. Two quick remarks:

(1) The above and below will continue to work in more generality, in particular, whenever the quotient structure maps q^* on $Y \in \mathsf{dSet}_G$ are *injections*.

(2) The "every" in Condition (2) refers to the choices of inclusion and quotient maps

$$G.K \cdot C_1 \hookrightarrow G/K \cdot (C_{[H:K]} \circ (C_1)) \xrightarrow{q} C_{H/K} \circ_K C_{K/K}$$

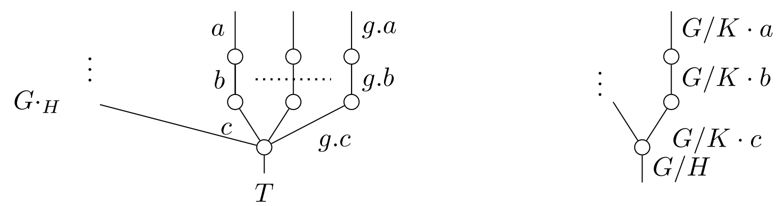
(where without loss of generality we have ignored the other orbits in A) By Lemma 6.3.31, this is equivalent to saying any associated orbital face $\partial_r\gamma \in i_*X(G \cdot C_1)$ is degenerate.

Lemmas 6.3 and 6.4 in [MW09] carry through to this setting, using Lemma 5.2.29.

Lemma 6.3.34 (cf. [MW09, Lemma 6.3]). *The relation $\sim_K$ is an equivalence relation on $X(C_A)$.*

Proof. To prove reflexivity, we just note that $\sigma_{G.e} f \in i_* X(C_A \circ_{K.e} C_{K/K})$ observes the homotopy $f \sim_K f$.

For symmetry, suppose we are given $f, \bar{f} \in X(C_A)$, and a homotopy $\gamma_f^{\bar{f}} \in X(C_A \circ_K C_{K/K})$. We will now build an orbital $G.c$-horn $\Lambda^{G.c}_{\mathrm{Orb}}[T]$ of $T = C_A \circ_{K.c} C_{K/K} \circ_{K.b} C_{K/K}$:



This consists of the compatible data for all of the orbital faces except $T \setminus G.c$ and T.

(1) for $T \setminus G.b$, we select $\sigma^* q^* \ f = q^* \sigma^* \ f \in X(C_{|A|} \circ_{\{K\}} (C_1))$;

(2) for $T \setminus G.v_a$: we select $q^* \ \gamma_f^{\bar{f}} \in X(C_{|A|} \circ_{\{K\}} (C_1))$; and

(3) for any root-cluster face map $C_{K/K} \circ_{K.c} C_{K/K}$, we select the degeneracy $\sigma^*(*) \in X(C_1 \circ C_1)$, with $\{*\} = X(\eta, *)$.

These are in fact compatible, and determine an orbital horn: we have

$$\partial_{\{H.v_a\}}(\sigma^* q^* f) = q^* f = \partial_{\{H.v_b\}}(q^* \gamma_f^{\bar{f}}) = q^* \partial_{G.v_b} \gamma_f^{\bar{f}},$$

and the two root-cluster faces factor through a degeneracy. Thus, by Lemma 5.2.29, there exists $\chi \in X(C_A \circ_{K.c} C_{K/K} \circ_{K.b} C_{K/K})$ such that $\bar{\gamma} := \partial_{G.c} \chi$ has the properties that

(1) the quotient of the inner face is $q^* f$;

$$q^*(\partial_{G.b} \bar{\gamma}) = \partial^*_{\{H.c\}} \sigma^* q * f = q^* f$$

(2) the quotient of the leaf-cluster outer face is $q^*\bar{f}$;

$$q^*(\partial_{G.v_a}\bar{\gamma}) = \partial_{\{H.c\}} q^*\gamma_f^{\bar{f}} = q^*\partial^*_{G.c}\gamma_f^{\bar{f}} = q^*\bar{f}$$

(3) each root cluster orbital face map is a degeneracy.

Since every q^* is injective, this implies that $\bar{\gamma}$ observe a homotopy $\bar{f} \sim_K f$, as desired.

Transitivity is proved analogously, by following the proof in [MW09]. $\square$

Similarly, we prove:

Lemma 6.3.35 (cf. [MW09, Lemma 6.4]). *The equivalence relations $\sim_{K_i}$ on $i_*X(C_A)$ are all equal.* $\square$

We denote the equivalence class of f by $[f]$.

Remark 6.3.36. Non-equivariantly, we note that if we have a dendrix $\gamma \in X(C_n \circ (C_1))$ such that the smallest outer face is f and the smallest inner face is $\bar{f}$, then $f \sim \bar{f}$. Indeed, if $\{a_1, \ldots, a_n\}$ denote the inner edges, we let $f_i = \partial_{a_1 \ldots, a_i}\partial_{v_{a_{i+1}}, \ldots, v_{a_n}}\gamma$. Then $\partial_{a_1, \ldots, a_i}\partial_{v_{a_{i+2}}, \ldots, v_{a_n}}\gamma$ observes a homotopy $f_i \sim f_{i+1}$. We say that γ observes an *iterated homotopy $f \sim \bar{f}$*.

Moreover, it is clear that the same story will hold over equivariantly.

We can now define the underlying sequence of our homotopy operad.

Definition 6.3.37. Given a G-∞-operad $X \in \mathsf{dSet}^G$ with $X(G \cdot_H \eta) = *$ for all $H \leq G$, define $\mathrm{Ho}(X)$ to be the G-symmetric sequence defined by $\mathrm{Ho}(X)(C) = i_*X(C)/\sim$.

The definition of composition of elements is again similar to that in [MW09]:

Definition 6.3.38. Given $A = \amalg H/K_i \amalg H/K$, $B \in \mathsf{Set}^K$, $f \in X(C_A)$, and $\bar{f} \in X(C_B)$, we say that p is a *composition* of f and $\bar{f}$ if there exists a dendrix $\gamma \in X(C_A \circ_{K.e} C_B)$ such that

- the leaf cluster outer G-face $\partial^*_{G.v_e}\gamma = f$; and
- the root cluster face $\partial^*_{G.v_r}\gamma = g$.
- the inner G-face $\partial^*_{G.e}\gamma = p$;

We say γ *observes* the composition, and write $\gamma : p \sim f \circ_K \bar{f}$.

Lemma 6.3.39 (cf. [MW09, Lemma 6.8])**.** *Given $\gamma : p \sim f \circ_K \bar{f}$ and $\gamma' : p' \sim f \circ_K \bar{f}$, then we must have $p \sim p$.*

Proof. Without loss of generality, we may assume $A = H/K$ and $B = K/L$. Let $T = C_{H/K} \circ_{K.c} C_{K/L} \circ_{L.b} C_{L/L}$. We consider the horn $\Lambda^{G.c}_{\mathrm{Orb}}[T]$ given by:

(1) for $T \setminus G.v_a$, we select the dendrix $q^*\gamma$;

(2) for $T \setminus G.b$, we select the dendrix $q^*\gamma'$; and

(3) for any root-cluster face $C_{K/L} \circ_{K.b} C_{L/L}$, we select the dendrix $q^*\gamma_{\bar{f}}$, the degenerate homotopy $\bar{f} \sim \bar{f}$.

Again, it is easy to check that these are compatible, and thus define an orbital horn; let $x \in i_*X(T)$ be a lift. The inner face $\chi := partial_{G.c}x$ has the properties that:

$$q^*(\partial_{G.b}\chi) = \partial_{\{H.e\}}q^*\gamma' = q^*\partial_{G.c}\gamma' = q^*p'$$
$$q^*(\partial_{G.v_a}\chi) = \partial_{\{H.e\}}q^*\gamma = q^*\partial_{G.c}\gamma = q^*p.$$

Finally, q^* injective implies χ observe a homotopy $p \sim p'$. □

An identical argument paralleling the original source (as the two proofs above demonstrate) yields the following.

Lemma 6.3.40 (cf. [MW09, Lemma 6.9])**.** *If $f \sim f'$ and $\bar{f} \sim \bar{f}'$, and $p \sim f \circ_K \bar{f}$ and $p' \sim f' \circ_K \bar{f}$, then $p \sim p'$.*

Combining the above two lemmas shows us that $\mathrm{Ho}(X)$ has a well-defined composition. Moreover, an analogous proof as in [MW09], replacing the use of Lemma 5.1 (*loc. cit*) with the above Lemma 5.2.34, shows that in fact this composition is associative and unital.

Proposition 6.3.41 (cf. [MW09, Proposition 6.6])**.** *The operation* $[f] \circ_H [g] = [f \circ_H g]$ *gives a well-defined map*

$$\mathrm{Ho}(X)(A) \times \mathrm{Ho}(X)(B) \to \mathrm{Ho}(X)(\partial_K(A \circ_K B)).$$

Moreover, this endows $\mathrm{Ho}(X)$ *with the structure of a (monadic) genuine* G*-operad.*

Proof. Since this composition is associative, we have a (unique) map

$$N_{\mathrm{Ho}(X)}(T) \to \mathrm{Ho}(X)(val(T)).$$

It suffices to check that this map is natural in quotient maps. By Remark 6.3.36, we observe that if γ observes a homotopy $f \sim_K \bar{f}$, then $q^*\gamma$ observes an iterated homotopy $q^*f \sim q^*\bar{f}$. Finally, if γ observes the composition $p : f \circ_K \bar{f}$, then $q^*\gamma$ will observe the iterated composition $q^*p : q^*f \circ (q^*\bar{f})$, where the q^* affects $\bar{f}$ will change, depending on the orbital structure of q^*f. As this is precisely what happens monadically, we are finished. $\square$

6.3.4 Comparison with G-Operads

We observed above that $i_*N\mathcal{O} \in \mathsf{dSet}_G$ for any operad $\mathcal{O}$ was a (sheaf) genuine G-operad. We can show this using a more algebraic comparison between the monads $\mathbb{F}$ and $\mathbb{F}_G$. Indeed, recall that we have an underlying adjunction $\mathsf{Sym}^G \leftrightarrows \mathsf{Sym}_G$.

Proposition 6.3.42. *The above adjunction lifts to an adjunction of operads*

$$\begin{tikzcd} \mathsf{Op}^G \arrow[r, bend left, "i_!"] & \mathsf{Op}_G \arrow[l, "i^*"'] \end{tikzcd} \quad \begin{tikzcd} \mathsf{Op}^G \arrow[r, bend right, "i_*"'] & \mathsf{Op}_G \end{tikzcd}$$

We prove this using a series of lemmas and propositions relating the structure monads of these categories. We begin by establishing some notation.

Definition 6.3.43. We identify the following unit and counit natural transformations from the above adjunction.

$$\eta_r : id \overset{\simeq}{\Rightarrow} i_!^* \qquad\qquad \eta_l : id \Rightarrow i_* i^*$$

$$\epsilon_r : i_! i^* \Rightarrow id \qquad\qquad \epsilon_l : i^* i_* \overset{\simeq}{\Rightarrow} id$$

Moreover, we let $\theta : i_! \Rightarrow i_*$ denote either equal composite

$$\begin{tikzcd}
i_! \arrow[r, Rightarrow, "\epsilon_l^{-1}"] \arrow[d, "\eta_l"'] & i_! i^* i_* \arrow[d, Rightarrow, "\epsilon_r"] \\
i_* i^* i_! \arrow[r, Rightarrow, "\eta_r^{-1}"] & i_*
\end{tikzcd}$$

As $i^*\mathbb{F}_G X$ only depends on i^*X, we have the following:

Lemma 6.3.44. *The three natural transformations below are invertible.*

$$i^*\mathbb{F}_G \cdot (\eta_l) : i^*\mathbb{F}_G \Rightarrow i^*\mathbb{F}_G i_* i^*$$

$$i^*\mathbb{F}_G \cdot (\epsilon_r) : i^*\mathbb{F}_G i_! i^* \Rightarrow i^*\mathbb{F}_G$$

$$i^*\mathbb{F}_G \cdot (\theta) : i^*\mathbb{F}_G i_! \Rightarrow i^*\mathbb{F}_G i_*$$

□

Corollary 6.3.45. *The following composite is the identity.*

$$i^*\mathbb{F}_G \xRightarrow{\eta_l} i^*\mathbb{F}_G i_* i^* \xRightarrow{\theta^{-1}} i^*\mathbb{F}_G i_! i^* \xRightarrow{\epsilon_r} i^*\mathbb{F}_G.$$

Proof. This follows by the construction of θ, and the fact that (η_l, ϵ_l) define an adjunction by unpacking definitions. □

Similarly, $\mathbb{F}_G i_! X$ is empty on non-free G-trees, yielding the following:

Lemma 6.3.46. $\epsilon_{\mathbb{F}_G i_!} : i_! i^* \mathbb{F}_G i_! \Rightarrow \mathbb{F}_G i_!$ *is a natural isomorphism.* $\square$

As a first step in our comparison, we make the following observation.

Proposition 6.3.47.

$$\mathbb{F} \simeq i^* \circ \mathbb{F}_G \circ i_!.$$

Proof. This result reduces to the claim that, for any $X \in \mathsf{Sym}_G$, the value of $\mathbb{F}_G A(G \cdot T_0)$ only depends on the evaluation of A on free G-corollas. This follows since the overcategory $\mathbb{T}_0^{op} \downarrow G \cdot T_0$ consists entirely of free G-trees, with free G-corollas as vertices. $\square$

With this identification, we have that the structure maps for $\mathbb{F}$ are given by the composite

$$i * \mathbb{F}_G i_! i^* \mathbb{F}_G i_! \overset{\epsilon_r}{\Rightarrow} i^* \mathbb{F}_G \mathbb{F}_G i_! \xrightarrow{\gamma} \Rightarrow i^* \mathbb{F}_G i_!$$

and either composite

$$\begin{tikzcd}
i^* i_! X \arrow[dr, "1_{\mathbb{F}_G}"'] & X \arrow[l, "\eta_r"'] \arrow[r, "\epsilon_l^{-1}"] \arrow[d, "1_{\mathbb{F}_G}"] & i^* i_* X \arrow[d, "1_{\mathbb{F}_G}"] \\
& i^* \mathbb{F}_G i_! X & i^* \mathbb{F}_G i_* X \arrow[l, "\theta^{-1}"]
\end{tikzcd}$$

where again $\gamma : \mathbb{F}_G \mathbb{F}_G \Rightarrow \mathbb{F}_G$ is the monadic structure map, and $1_{\mathbb{F}_G}$ is the unit transformation of $\mathbb{F}_G$.

proof of Proposition 6.3.42. We will show that these functors lift to functors between operadic categories, and then show that the adjunction units and counits are maps of operads.

First, let $X \in \mathsf{Op}^G$, and consider $i_* X \in \mathsf{Sym}_G$. We define the $\mathbb{F}_G$ structure map on $i_* X$ as the composite

$$\mu : \mathbb{F}_G i_* X \xrightarrow{\eta_l} i_* i^* \mathbb{F}_G i_* X \xrightarrow{\theta^{-1}} i_* i^* \mathbb{F}_G i_! X \xrightarrow{\mu} i_* X$$

where μ also refers to the structure map $\mathbb{F}X = i^* \mathbb{F}_G i_! X \to X$. This map is associative, as demonstrated by the following commuting diagram:

$$
\begin{tikzcd}
\mathbb{F}_G\mathbb{F}_G i_* X \arrow[r, "\eta_l"] \arrow[d, "\eta_l"'] \arrow[ddd, bend right, equal] & \mathbb{F}_G i_* i^* \mathbb{F}_G i_* X \arrow[r, "\theta^{-1}"] \arrow[d, "\eta_l"'] & \mathbb{F}_G i_* i^* \mathbb{F}_G i_! X \arrow[r, "\mu"] \arrow[d, "\eta_l"'] & \mathbb{F}_G i_* X \arrow[d, "\eta_l"] \\
i_* i^* \mathbb{F}_G \mathbb{F}_G i_* X \arrow[r, "\eta_l"] & i_* i^* \mathbb{F}_G i_* i^* \mathbb{F}_G i_* X \arrow[r, "\theta^{-1}"] \arrow[d, "\theta^{-1}"'] & i_* i^* \mathbb{F}_G i_* i^* \mathbb{F}_G i_! X \arrow[r, "\mu"] \arrow[d, "\theta^{-1}"'] & i_* i^* \mathbb{F}_G i_* X \arrow[d, "\theta^{-1}"] \\
 & i_* i^* \mathbb{F}_G i_! i^* \mathbb{F}_G i_* X \arrow[r, "\theta^{-1}"] \arrow[d, "\epsilon_r"'] & i_* i^* \mathbb{F}_G i_! i^* \mathbb{F}_G i_! X \arrow[r, "\mu"] \arrow[d, "\epsilon_r"'] & i_* i^* \mathbb{F}_G i_! X \arrow[dd, "\mu"] \\
\mathbb{F}_G\mathbb{F}_G i_* X \arrow[r, "\eta_l"] \arrow[uu, dotted, "\eta_l"] \arrow[d, "\gamma"'] & i_* i^* \mathbb{F}_G \mathbb{F}_G i_* X \arrow[r, "\theta^{-1}"] \arrow[d, "\gamma"'] & i_* i^* \mathbb{F}_G \mathbb{F}_G i_! X \arrow[d, "\gamma"'] & \\
\mathbb{F}_G i_* X \arrow[r, "\eta_l"] & i_* i^* \mathbb{F}_G i_* X \arrow[r, "\theta^{-1}"] & i_* i^* \mathbb{F}_G i_! X \arrow[r, "\mu"] & i_* X
\end{tikzcd}
$$

The bottom right large square commutes as the $\mathbb{F}$-action on X is associative, the left large square commutes via the above Corollary, and the rest commute by naturality.

For unitality, we have the following commuting diagram, where the bottom triangle encodes the unitality of X as an $\mathbb{F}$-algebra, and the rest commutes by naturality or by definition.

$$
\begin{tikzcd}
i_* X \arrow[rr, "1_{\mathbb{F}_G}"] \arrow[dr, "\eta_l"] \arrow[dd, equal] & & \mathbb{F}_G i_* X \arrow[d, "\eta_l"] \\
 & i_* i^* i_* X \arrow[r, "1_{\mathbb{F}_G}"] \arrow[dl, "\epsilon_l"'] & i_! i^* \mathbb{F}_G i_* X \arrow[d, "\theta^{-1}"] \\
i_* X \arrow[r, "\eta_r"] \arrow[drr, equal] & i_* i^* i_! \arrow[r, "1_{\mathbb{F}_G}"] \arrow[u, "\theta"] & i_! i^* \mathbb{F}_G i_! X \arrow[d, "\mu"] \\
 & & i_* X
\end{tikzcd}
$$

Now we consider $i_! X$, and define it's $\mathbb{F}_G$-structure map as the composite

$$
\mu : \mathbb{F}_G i_! X \xrightarrow{\epsilon_r^{-1}} i_! i^* \mathbb{F}_G i_! X \xrightarrow{\mu} i_! X,
$$

This structure is associative and unital by noting that the following diagrams commute:

$$
\begin{tikzcd}
\mathbb{F}_G\mathbb{F}_G i_! X \arrow[r, "\epsilon_r^{-1}"] \arrow[d, "\epsilon_r^{-1}"'] & \mathbb{F}_G i_! i^* \mathbb{F}_G i_! X \arrow[r, "\mu"] \arrow[d, "\epsilon_r^{-1}"] & \mathbb{F}_G i_! X \arrow[d, "\epsilon_r^{-1}"] \\
i_! i^* \mathbb{F}_G \mathbb{F}_G i_! X \arrow[r, "\epsilon_r^{-1}"] \arrow[d, "\epsilon_r"'] & i_! i^* \mathbb{F}_G i_! i^* \mathbb{F}_G i_! X \arrow[r, "\mu"] \arrow[d, "\epsilon_r"] & i_! i^* \mathbb{F}_G i_! X \arrow[dd, "\mu"] \\
\mathbb{F}_G\mathbb{F}_G i_! X \arrow[r, "\epsilon_r^{-1}"] \arrow[d, "\gamma"'] & i_! i^* \mathbb{F}_G \mathbb{F}_G i_! X \arrow[d, "\gamma"'] & \\
\mathbb{F}_G i_! X \arrow[r, "\epsilon_r^{-1}"] & i_! i^* \mathbb{F}_G i_! X \arrow[r, "\mu"] & i_! X
\end{tikzcd}
\qquad
\begin{tikzcd}
i_! X \arrow[r, equal] \arrow[ddrr, equal, bend right] & i_! X \arrow[r, "1_{\mathbb{F}_G}"] \arrow[d, "\eta_r"'] & \mathbb{F}_G i_! X \arrow[d, "\epsilon_r^{-1}"] \\
& i_! i^* i_! X \arrow[r, "1_{\mathbb{F}_G}"] & i_! i^* \mathbb{F}_G i_! X \arrow[d, "\mu"] \\
& & X
\end{tikzcd}
$$

where the bottom right of the left diagram, and the left stairwell of the right diagram, commute since $\mu : \mathbb{F} \to X$ is associative and unital.

Finally, given $X \in \mathsf{dSet}_G$, consider $i^* X$. This has an $\mathbb{F}$-algebra structure given my the composite

$$\mu : \mathbb{F} i^* X = i^* \mathbb{F}_G i_! i^* X \xrightarrow{\epsilon_r} i^* \mathbb{F}_G X \xrightarrow{\mu} i^* X.$$

Associativity and unitality are confirmed by the following commuting diagrams.

$$
\begin{tikzcd}
i^* \mathbb{F}_G i_! i^* \mathbb{F}_G i_! i^* X \arrow[r, "\epsilon_r"] \arrow[d, "\epsilon_r"'] & i^* \mathbb{F}_G i_! i^* \mathbb{F}_G X \arrow[r, "\mu"] \arrow[d, "\epsilon_r"'] & i^* \mathbb{F}_G i_! i^* X \arrow[d, "\epsilon_r"] \\
i^* \mathbb{F}_G \mathbb{F}_G i_! i^* X \arrow[r, "\epsilon_r"] \arrow[d, "\gamma"'] & i^* \mathbb{F}_G \mathbb{F}_G X \arrow[r, "\mu"] \arrow[d, "\gamma"'] & i^* \mathbb{F}_G X \arrow[d, "\mu"] \\
i^* \mathbb{F}_G i_! i^* X \arrow[r, "\epsilon_r"] & i^* \mathbb{F}_G X \arrow[r, "\mu"] & i^* X
\end{tikzcd}
\qquad
\begin{tikzcd}
i^* X \arrow[r, "\eta_r"] \arrow[dr, equal] & i^* i_! i^* X \arrow[r, "1_{\mathbb{F}_G}"] \arrow[d, "\epsilon_r"] & i^* \mathbb{F}_G i_! i^* X \arrow[d, "\epsilon_r"] \\
& i^* X \arrow[r, "1_{\mathbb{F}_G}"] \arrow[dr, equal] & i^* \mathbb{F}_G X \arrow[d, "\mu"] \\
& & i^* X
\end{tikzcd}
$$

Lastly, we show that the four unit and counit maps are given by maps of operads. For η_r and ϵ_l, this is immediate as they are isomorphisms on sequences. For η_l (respectively

ϵ_r), consider the left (resp. right) diagram below.

$$\begin{tikzcd}
\mathbb{F}_G X \arrow[r, "\eta_l"] \arrow[d, "\eta_l"'] & \mathbb{F}_G i_* i^* X \arrow[d, "\eta_l"] \\
i_* i^* \mathbb{F}_G X \arrow[r, "\eta_l"] & i_8 i^* \mathbb{F}_G i_* i^* X \arrow[d, "\theta^{-1}"] \\
 & i_* i^* \mathbb{F}_G i_! i^* X \arrow[d, "\epsilon_r"] \arrow[dd, bend left, "\mu_{i^* X}"] \\
\mathbb{F}_G X \arrow[uu, "\eta_l"] \arrow[r, "\eta_l"] \arrow[d, "\mu"'] & i_* i^* \mathbb{F}_G X \arrow[d, "\mu"] \\
X \arrow[r, "\eta_l"] & i_* i^* X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{F}_G i_! i^* X \arrow[r, "\epsilon_r"] \arrow[d, "\epsilon_r^{-1}"'] & \mathbb{F}_G X \arrow[d, "\epsilon_r^{-1}"] \arrow[dd, bend left, equal] \\
i_! i^* \mathbb{F}_G i_! i^* X \arrow[r, "\epsilon_r"] \arrow[d, "\epsilon_r"] \arrow[dd, bend right, "\mu_{i^* X}"'] & \mathbb{F}_G i_! i^* X \arrow[d, "\epsilon_r"] \\
i_! i^* \mathbb{F}_G X \arrow[r, "\epsilon_r"] \arrow[d, "\mu"] & \mathbb{F}_G X \arrow[d, "\mu"] \\
i_! i^* X \arrow[r, "\epsilon_r"] & X
\end{tikzcd}$$

where each square commutes either by the naturality of η_l" (resp. ϵ_r) or by the definition of θ^{-1}.

This finishes the proof. $\square$

This also provides a solution to one of our motivating issues:

Example 6.3.48. As a particular example of the above, we have a map $i_* : \mathsf{sOp}^G \to \mathsf{sOp}_G$ with $i_*\mathcal{O}(G \cdot C_n/N) := \mathcal{O}(n)^N$. Moreover, we can post-compose with π_0 levelwise, yielding a *non-commutative* diagram

$$\begin{tikzcd}
\mathsf{Op}^G \arrow[d, "i_*"] & \mathsf{sOp}^G \arrow[l, "\pi_0"'] \arrow[d, "i_*"] \\
\mathsf{Op}_G & \mathsf{sOp}_G \arrow[l, "\pi_0"]
\end{tikzcd}$$

6.4 Arrow Composition Product Model

In the monadic model above, we generalized the free operad monad of Section 3.2, and found that any reasonable equivariant generalization yielded an algebraic structure with significantly more data than regular equivariant operads. In this section, we generalize the coend description of the composition product. In Section 4.2.3, we referenced a new presentation of this product by changing the underlying symmetric sequence to an equivalent but more expressive category. This allowed us to present more of the data which is already present in G-operads. We will now again update this indexing category, but this time to a much larger and more interested category, capturing more information. Again, the underlying principal is exploit a different presentation of G-corollas.

Definition 6.4.1. Let Pull_G be the subcategory of the category of arrows $\mathrm{Arrow}(\mathsf{F}^G)$, where morphisms between arrows are only allowed if they are pullbacks. More generally, if $\mathcal{F}$ is a (strong) indexing system, let $\mathsf{Pull}_{\mathcal{F}}$ be the analogous subcategory of arrows in $\mathsf{F}^{\mathcal{F}}$, where $\mathsf{F}^{\mathcal{F}}$ is the the wide subcategory of F^G where $f : S \to T$ is in $\mathsf{F}^{\mathcal{F}}$ if and only if for all $s \in S$, $\mathrm{Stab}_G(f(s)).s$ is $\mathcal{F}$-admissible (see [BH16, Theorem 3.10], where they show that $\mathsf{F}^{\mathcal{F}}$, notated $\mathsf{Set}^G_{\mathcal{O}}$, is in fact a category).

These all have an obvious symmetric monoidal product given by the coproduct of maps.

Lemma 6.4.2. Pull_G *is equivalent to the coproduct completion of* Σ_G*. Thus, the category of G-symmetric sequences is equivalent to the category* $\mathrm{Fun}^\times(\mathsf{Pull}_G, \mathcal{V})$ *of functors sending coproducts to products.*

Proof. Any arrow in F^G has a decomposition $\amalg_i(A_i \to B_i)$ with $A, B \in \mathsf{Set}^G$ and the B_i transitive (and the A_i possibly empty). Any map to a transitive G-set can be represented as a G-corolla, so this map can be represented as a coproduct of corollas. In both situations, the maps allowed are just quotient maps, as desired. □

Definition 6.4.3. Let Pull^1_G denote the pullback category

$$\begin{tikzcd} \mathsf{Pull}^1_G \arrow[r] \arrow[d] & \mathsf{Pull}_G \arrow[d, "s"] \\ \mathsf{Pull}_G \arrow[r, "t"] & \mathsf{F}^G \end{tikzcd}$$

where s and t are the "source" and "target" maps. Explicitly, objects are pairs (f_1, f_2) of composable arrows in F^G, and objects are stacked squares which are each individually pullbacks.

Definition 6.4.4. Given $X, Y \in \mathrm{Fun}^\times(\mathsf{Pull}_G, \mathcal{V})$, define the *composition product* to be the coend

$$X \circ Y(-) = \int^{(f_1,f_2)\in \mathsf{Pull}_G^1} X(f_1) \times Y(f_2) \times \mathsf{Pull}_G(-, f_2 f_1).$$

Proposition 6.4.5. *The composition product induces a monoidal product on the category* $\mathrm{Fun}^{\times}(\mathsf{Pull}_G, \mathcal{V})$.

Proof. An easy calculation with Yoneda reduction (Lemma 1.1.8) shows that iterated composition products are given by

$$X_1 \circ \ldots \circ X_m \simeq \int^{f_1,\ldots,f_m} X_1(f_1) \otimes \ldots \otimes X_m(f_m) \cdot \mathsf{Pull}_G(-, f_m \ldots, f_2 f_1).$$

so the operation is clearly associative. The unit J is given by

$$J(-) = \mathsf{Pull}_G(-, id_{G/G}),$$

as can be verified easily, again using Yoneda reduction. □

Definition 6.4.6. A *(coend) genuine G-operad* is a monoid $\mathcal{O}$ in $(\mathrm{Fun}^{\times}(\mathsf{Pull}_G, \mathcal{V}), \circ)$.

Definition 6.4.7. We say $X \in \mathrm{Fun}^{\times}(\mathsf{Pull}_G, \mathcal{V})$ is *concentrated in degree 0* if $X(f) = \varnothing$ for all maps with source not equal to the empty set.

Definition 6.4.8. Given $X \in \mathrm{Fun}^{\times}(\mathsf{Pull}_G, \mathcal{V})$ and a genuine G-operad $\mathcal{O}$, an *$\mathcal{O}$-module* structure on X is a module over $\mathcal{O}$ is the monoidal category under $\circ$. That is, we have a map $\mathcal{O} \circ X \to X$ which is unital and associative. If the module X is concentrated in degree 0, we call it an *algebra*.

In this description, since we are just considering monoids in a monoidal category, there is a lot of straightforward category theory to work with about algebras and modules. However, putting on model structure on this version would be quite difficult, as the monoidal product we are working with is not particularly conducive to analysis. However, we have a similar-looking conjecture:

Conjecture 6.4.9. *For any (strong) indexing system $\bar{\mathcal{F}} \subseteq \mathcal{F}$, the category of (coend) genuine $\mathcal{F}$-operads has the $\bar{\mathcal{F}}$-projective model structure induced by the forgetful functor*

$$\mathsf{Op}_{\mathcal{F}} \to \mathsf{Sym}_{\mathcal{F}} \to \mathsf{Sym}_{\bar{\mathcal{F}}},$$

where $\varphi : X \to Y$ is a weak equivalence (respectively, fibration) if for all $\bar{\mathcal{F}}$-admissible arrows f in $\mathsf{F}^{\bar{\mathcal{F}}}$, $\varphi(f) : X(f) \to Y(f)$ is so.

Additionally, there should be a model of coend genuine equivariant operads for any *weak* indexing system $\mathcal{F}$; however, it is not currently understood what the category $\mathsf{F}^{\mathcal{F}}$ should be in this case. If such a structure does not exist, there should be a proper reason why not. This too will be the subject of further research.

6.5 Composition Product Model

In this section, we present our last model of genuine G-operads, during which again we restrict to the single-coloured case. We continue to generalize the composition product, but instead this time build off of the version of the composition product given in Section 3.6, again replacing the use of trees with G-trees. As for the monad above, this natural generalizes to a algebraic structure which is richer than G-operads.

We ignoring the tensor product (which can be defined analogously), we move straight to defining the composition product:

Definition 6.5.1. Let $\mathbb{T}_{G,0}\{2\}$ denote the full subcategory of $\mathbb{T}_{G,0}$ of height-2 G-trees; that is, those trees T with a decomposition $T = C_A \circ (C_{B_{G.e}})$ with $B_{G.e}$ some $\mathrm{Stab}_G(e)$-set.

Given G-symmetric sequences X and Y in $\mathcal{V}^{\Sigma_G^{op}}$, we have a nerve-evaluation map

$$N_{(Y),X}\{2\} : \mathbb{T}_{G,0}\{2\}^{op} \xrightarrow{\mathbb{V}} (\mathsf{F} \wr \Sigma_G)^{op} \xrightarrow{\mathsf{F} \wr Y \times X} (\mathsf{F} \wr \mathcal{V}^{op})^{op} \xrightarrow{\times} \mathcal{V}$$

sending a tree $T = C_A \circ (C_{B_{G.e}})$ to $X(C_A) \times \Pi Y(C_{B_{G.e}})$.

Definition 6.5.2. The *composition product* $X \circ Y$ of G-symmetric sequences is defined to be the left Kan extension $\mathrm{Lan}_{val} N_{(Y),X}\{2\}$;

$$\begin{tikzcd}
\mathbb{T}_{G,0}\{2\}^{op} \arrow[r, "N_{(Y),X}\{2\}"] \arrow[d, "val"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\
\Sigma_G^{op} \arrow[ur, "\mathrm{Lan} \simeq X\circ Y"'] &
\end{tikzcd}$$

Conjecture 6.5.3. *The above composition product is a (non-symmetric) monoidal product on the category of G-symmetric sequences, with unit $\Sigma_G(-, C_{G/G})$.*

Definition 6.5.4. A *(composition) genuine G-operad* is a monoid under this composition product.

Conjecture 6.5.5. *The categories of monoids under the composition product in Sym_G is equivalent to the category of $\mathbb{F}_G$-algebras in Sym_G.*

This will be analogous to the similar comparison in the non-equivariant case, between the usual composition product description and our $\mathbb{F}$-algebra model.

6.6 Conjectured Model Structure

Given any weak indexing systems $\mathcal{F} \subseteq \bar{\mathcal{F}}$, the monadic and composition product models for genuine $\bar{\mathcal{F}}$-operads provide a natural candidate for an $\mathcal{F}$-model structure:

Definition 6.6.1. The $\mathcal{F}$-model structure, if it exists, is the projective model structure along the forgetful functor

$$\mathsf{Op}_{\bar{\mathcal{F}}} \to \mathcal{V}^{\Sigma_{\bar{\mathcal{F}}}} \to \mathcal{V}^{\Sigma}_{\mathcal{F}} \to \prod_{C \in \Sigma_{\mathcal{F}}} \mathcal{V}.$$

That is, $f : \mathcal{O} \to \mathcal{P}$ is a weak equivalence (respectively, fibration) if $f(C) : \mathcal{O}(C) \to \mathcal{P}(C)$ is so in $\mathcal{V}$ for all $\mathcal{F}$-admissible G-corollas C.

This follows the idea that Op_G is playing the role of "coefficient systems", and as such should have the projective model structure like $\mathsf{Top}^{O_G^{op}}$.

We expect to be able to extend the results from Sections 3.5, 4.3, and 5.3 into the (monadic) genuine-equivariant context. This would provide proofs of the following conjectures:

Conjecture 6.6.2. *Suppose $\mathcal{V}$ satisfies* Assumption 1*, and let $\mathcal{F} \subseteq \bar{\mathcal{F}}$ be weak indexing systems. Then $\mathcal{V}\mathsf{Op}_{\bar{\mathcal{F}}}^{\{*\}}$ has the $\mathcal{F}$-semi-model structure.*

Conjecture 6.6.3. *For any weak indexing system $\mathcal{F}$, the $\mathcal{F}$-model structure on $\mathcal{V}\mathsf{Op}_G$ exists if and only if it exists on $\mathcal{V}\mathsf{Op}_{\{*\}}^G$.*

Furthering the parallels with $\mathsf{Top}^G \leftrightarrows \mathsf{Top}^{O_G^{op}}$, we believe the following holds:

Conjecture 6.6.4. *[cf. Elmendorf's Theorem] $X \in \mathsf{Op}_G$ is cofibrant if and only if $X \simeq i_*Y$ for some $Y \in \mathsf{Op}^G$.*

Moreover, an immediate corollary of 6.6.4 would be that our models of G-operads are Quillen equivalent.

Corollary 6.6.5. *If Conjecture 6.6.4 holds and the $\mathcal{F}$-model structures exist on $\mathcal{V}\mathsf{Op}_{\{*\}}^G$ and $\mathcal{V}\mathsf{Op}_G$, then the adjunction*

$$\mathsf{Op}^G \underset{i_*}{\overset{i^*}{\leftrightarrows}} \mathsf{Op}_G$$

is a Quillen equivalence.

Proof. It is immediate that i_* would preserve (trivial) fibrations, and thus these form a Quillen pair. Moreover, 6.6.4 would imply that the cofibrant replacement for $i_*\mathcal{O}$ could be chosen to be equal to $i_*\mathcal{O}$, and hence the composite $i^*((i_*\mathcal{O})_{\mathrm{cof}}) \to \mathcal{O}$ would be the identity, thus an $\mathcal{F}$-equivalence in $\mathcal{V}\mathsf{Op}_{\{*\}}^G$ as desired. $\square$

Additionally, this would provide another proof of the N_∞-realization Conjecture 6.3.25: taking a cofibrant replacement $CN_{\mathcal{F}}$ of $N_{\mathcal{F}}$, we would then have that $i^*CN_{\mathcal{F}} \simeq N^{\mathcal{F}}$ in Op^G.

Appendix A

Kan Extensions

We collect some technical results about the naturality of Kan extensions on their input data, and their preservation under certain categorical constructions.

Remark A.0.1. For all of the results below, their formal dual result is true of *right* Kan extensions.

We begin with an easy result about "stacking" Kan extensions.

Lemma A.0.2. *Suppose we have functors*

$$\begin{tikzcd} \mathcal{C}' \arrow[d, "i"'] \arrow[dr, "X"] & \\ \mathcal{C} \arrow[r, "Y"] \arrow[d, "j"'] & \mathcal{V} \\ \mathcal{D} & \end{tikzcd}$$

such that $Y = \operatorname{Lan}_i X$*. Then* $\operatorname{Lan}_{ji} X \simeq \operatorname{Lan}_j Y$*.*

Proof. This follows from the Yoneda Lemma by directly unpacking the universal properties of the two functors. $\square$

A.1 Naturality

We formulate precisely what data Kan extensions are natural over.

Definition A.1.1. Let $\mathsf{WSpan}(\mathcal{D}, \mathcal{V})$ be the following category. Objects are spans of functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{V} \\
\mathcal{D} &
\end{tikzcd}$$

while morphisms are pairs (F, Φ)

$$\begin{tikzcd}
\mathcal{C}' \arrow[d, "F"'] \arrow[dr, bend left] \arrow[dd, bend right=60] & \\
\mathcal{C} \arrow[r] \arrow[d] \arrow[ur, phantom, "\Phi\,\Swarrow"] & \mathcal{V} \\
\Sigma &
\end{tikzcd}$$

with F a functor such that the left triangle commutes, and Φ a natural transformation.

Proposition A.1.2. *The left Kan extension operation is a functor* $\mathsf{WSpan}(\mathcal{D}, \mathcal{V}) \to \mathcal{V}^{\mathcal{D}}$*.*

Proof. This is a straightforward diagram chase via the universal property of the left Kan extension. Indeed, suppose we are given the following data:

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "i"'] \arrow[dr, "X"] & \\
\mathcal{C}' \arrow[r, "Y"] \arrow[d, "j"'] \arrow[ur, phantom, "\Phi\,\Swarrow"] & \mathcal{V} \\
\mathcal{D} \arrow[ur, dotted] &
\end{tikzcd}$$

Then, since we always have a natural transformation $\mathrm{Lan}_i\, Y \circ i \Rightarrow Y$, we have the following chain of bijections and maps:

$$\begin{aligned}
\mathcal{V}^{\mathcal{D}}(\mathrm{Lan}_j\, Y, \mathrm{Lan}_j\, Y) = \mathcal{V}^{\mathcal{C}'}(Y, \mathrm{Lan}_j\, Y \circ j) \xrightarrow{(\tilde{id})^*} \mathcal{V}^{\mathcal{C}'}(\mathrm{Lan}_i(Yi), \mathrm{Lan}_j\, Y \circ j) = \mathcal{V}^{\mathcal{C}}(Yi, \mathrm{Lan}_j\, Y \circ ji) \\
\xrightarrow{\Phi^*} \mathcal{V}^{\mathcal{C}}(X, \mathrm{Lan}_j\, Y \circ ji) = \mathcal{V}^{\mathcal{D}}(\mathrm{Lan}_{ji}\, X, \mathrm{Lan}_j\, Y).
\end{aligned}$$

The image of the identity, denoted Φ_*, is the desired natural transformation. It can similarly be shown that this process preserves compositions. $\square$

Diagrammatically, $\Phi_* : \operatorname{Lan}_{ji} X \to \operatorname{Lan}_j Y$ is the unique map such that the diagram below commutes:

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[dr, "\alpha_X"'] & Y \circ i \arrow[r, "\alpha_Y \circ i"] & \operatorname{Lan}_j Y \circ ji \\ & \operatorname{Lan}_{ji} X \circ ji \arrow[ur, "\Phi_* \circ ji"'] & \end{tikzcd} \tag{A.1}$$

We highlight a special case:

Corollary A.1.3. *Suppose we have functors $\mathcal{C} \xrightarrow{i} \mathcal{C}' \xrightarrow{j} \mathcal{D}$ and $X : \mathcal{C}' \to \mathcal{V}$. Then we have a natural transformation $\Phi : \operatorname{Lan}_{ji} Xi \to \operatorname{Lan}_j X$.*

$$\begin{tikzcd} \mathcal{C} \arrow[d, "i"'] \arrow[dr, "Xi"] & \\ \mathcal{C}' \arrow[r, "X"] \arrow[d, "j"'] & \mathcal{V} \\ \mathcal{D} \arrow[ur, dotted, shift left] \arrow[ur, dotted, shift right] & \end{tikzcd}$$

□

A.1.1 Left Kan Extensions and Pushouts

While dealing with general pushouts of categories requires solving a "word problem" on morphisms, there is a stronger notion which is much easier to understand. We recall that, given a square of categories

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{C}' \arrow[d] \\ \mathcal{B} \arrow[r] & \mathcal{C} \end{tikzcd}$$

if the nerve of this square is a pushout in sSet, then the above is a pushout of categories (since the nerve is the inclusion of a reflective subcategory). The pushouts that most concern us in this thesis are of this form.

Definition A.1.4. We call such squares *nervous pushouts* of categories.

If we further assume that the span of functors is built out of fully-faithful inclusions, these pushouts behave as nicely as possible with respect to left Kan extensions.

Lemma A.1.5. *Giveny any diagram in categories of the form*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{C}' \arrow[d, "i"] & \\
\mathcal{B} \arrow[r, "g"] & \mathcal{C} \arrow[r, "Y"] \arrow[d, "j"] & \mathcal{V} \\
& \mathcal{D} &
\end{tikzcd}$$

such that the square is a nervous pushout of fully-faithful functors, then $\mathrm{Lan}_j Y$ *is the pushout of the induced span*

$$\begin{tikzcd}
\mathrm{Lan}_{jif}(Yif) \arrow[r] \arrow[d] & \mathrm{Lan}_{ji}(Yi) \\
\mathrm{Lan}_{jg}(Yg). &
\end{tikzcd}$$

Proof. By the universal property of left Kan extensions, it suffices to show that, for any functor $Z : \mathcal{V} \to \mathcal{D}$, the natural map

$$\mathcal{V}^{\mathcal{D}}(Y, Zj) \longrightarrow \mathcal{V}^{\mathcal{B}}(Yg, Zjg) \prod_{\mathcal{V}^{\mathcal{A}}(Yif, Zjif)} \mathcal{V}^{\mathcal{C}}(Yi, Zji)$$

is a bijection. These two sets give the same data: a collection of maps $\Phi_b : Y(b) \to Z(b)$ and $\Phi_c : Y(c) \to Z(c)$ for all $b \in \mathcal{B}$ and $c \in \mathcal{C}$, such that $\Phi_b = \Phi_c$ whenever $b = c \in \mathcal{A}$. In general, the compatibilites required on the right are less demanding. However, with the above assumptions, a map $d \to d'$ in $\mathcal{D}$ is *uniquely* a map in $\mathcal{A}$, $\mathcal{B} \setminus \mathcal{A}$, or $\mathcal{C} \setminus \mathcal{A}$, and thus all the necessary compatibilities are covered by (at least) one of the $\{\Phi_b\}$ or $\{\Phi_c\}$. $\square$

The following result will also be useful to our analysis.

Lemma A.1.6. *Suppose we have the following commutative diagram of natural transfor-*

mations

$$\begin{tikzcd}
 & \mathcal{C}' \arrow[rr] \arrow[dd] & & \mathcal{V} \arrow[dd, equal] \\
\mathcal{E}' \arrow[ur] \arrow[rr] \arrow[dd] \arrow[r, phantom, "\Nearrow"] & & \mathcal{V} \arrow[ur, equal] \arrow[dd, equal] & \\
 & \mathcal{C} \arrow[rr] \arrow[dd] & & \mathcal{V} \\
\mathcal{E} \arrow[ur] \arrow[rr] \arrow[r, phantom, "\Nearrow"] & & \mathcal{V} \arrow[ur, equal] & \\
 & \mathcal{D} & &
\end{tikzcd}$$

such that the left and right faces commute, the front and back faces are some natural transformations Φ *and* Ψ*, and the top and bottom faces are left Kan extension. Then the maps induced by Lemma A.1.2 from* Φ *and* Ψ *are isomorphic.*

Proof. We know the sources and targets are isomorphic by Lemma A.0.2. The result then follows as in the proof of Lemma A.1.2. $\square$

A.1.2 A Universal Property

Now, suppose we are given parallel maps $d_0, d_1 : X \to Y$ in $\mathsf{WSpan}(\mathcal{C}, \mathcal{V})$

$$\begin{tikzcd}
\mathcal{C}' \arrow[d, shift right, "\pi_0"'] \arrow[d, shift left, "\pi_1"] \arrow[dr, "X"] & \\
\mathcal{C} \arrow[r, "Y"] \arrow[d, "j"'] & \mathcal{V} \\
\mathcal{D} &
\end{tikzcd}$$

with $d_i = (\pi_i, d_i')$. By Corollary A.1.2, this induces a pair of parallel maps

$$\operatorname{Lan}_{ji} X \underset{(d_1)_*}{\overset{(d_0)_*}{\rightrightarrows}} \operatorname{Lan}_j Y$$

We will now describe a universal property of the coequalizer of these two maps:

Lemma A.1.7. *Let* $Q = \operatorname{coeq}((d_0)_*, (d_1)_*)$*. Then, for any* $Z : \mathcal{D} \to \mathcal{V}$*, the set of natural* *transformations* $\mathcal{V}^{\mathcal{D}}(Q,Z)$ *is in bijection with the set of functors* $F : Y \to Z \circ j$ *such that the diagram*

$$\begin{tikzcd} & Y \circ d_0 \arrow[dr, Rightarrow, "F \circ d_0"] & \\ X \arrow[ur, Rightarrow, "d_0"] \arrow[dr, Rightarrow, "d_1"'] & & Zji \\ & Y \circ d_1 \arrow[ur, Rightarrow, "F \circ d_1"'] & \end{tikzcd}$$

commutes.

Proof. By definition, we have that

$$\begin{aligned} \mathcal{V}^{\mathcal{D}}(Q,Z) &\simeq \mathcal{V}^{\mathcal{D}}(\operatorname{Lan}_j Y, Z) \prod_{\mathcal{V}^{\mathcal{D}}(\operatorname{Lan}_{ji} X, Z) \times \mathcal{V}^{\mathcal{D}}(\operatorname{Lan}_j Y, Z)} \mathcal{V}^{\mathcal{D}}(\operatorname{Lan}_j Y, Z) \\ &\simeq \mathcal{V}^{\mathcal{C}}(Y, Zj) \prod_{\mathcal{V}^{\mathcal{C}'}(X, Zji) \times \mathcal{V}^{\mathcal{C}}(Y, Z_j)} \mathcal{V}^{\mathcal{C}}(Y, Z_j) \end{aligned}$$

where the maps in the first pullback are $((d_0)_*)^* \times id$ and $((d_1)_*)^* \times id$, and in the second these maps are surrounded by adjoint isomorphisms. These latter compositions can then be identified, as desired, with $(d_0)^* \circ (\pi_0)^*(-)$ and $(d_1)^* \circ (\pi_1)^*(-)$, where $(\pi_i)^*(-) : \mathcal{V}^{\mathcal{C}} \to \mathcal{V}^{\mathcal{C}'}$ acts as precomposition on both functors; this follows from the factorization description of the $(d_i)_*$ from Diagram (A.1) and the fact that any map $F : Y \to Z \circ j$ must factor through α_Y. $\square$

A.1.3 Left Kan Extensions and Pullbacks

Recall the construction $\mathsf{F} \wr \mathcal{C}$ of Definition 3.1.2. We will show that this operation is compatible in many senses with left Kan extensions.

Lemma A.1.8. *Given a functor* $Y : \mathcal{E} \to \mathcal{V}$ *and the following diagrams of categories*

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "X"] \arrow[d, "i"'] \arrow[dr, phantom, "\Rrightarrow"] & \mathcal{V} \\ \mathcal{D}^{op} \arrow[ur, "L"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \times (\mathsf{F} \wr \mathcal{C}^{op})^{op} \arrow[r, "Y \times \mathsf{F} \wr X"] \arrow[d, "id \times \mathsf{F} \wr i"'] & \mathcal{V} \times (\mathsf{F} \wr \mathcal{V}^{op})^{op} \arrow[r, "\otimes"] & \mathcal{V} \\ \mathcal{E} \times (\mathsf{F} \wr \mathcal{D}^{op})^{op} \arrow[ur, "Y \times \mathsf{F} \wr R"'] \arrow[urr, dashed] & & \end{tikzcd}$$

where $L := \mathrm{Lan}_i X$ and $\mathcal{V}$ is a cocomplete symmetric monoidal category such that the monoidal product $\otimes$ commutes with colimits, we have that the dashed arrow in the second diagram is the left Kan extension of the top row over $id \times \mathsf{F} \wr i$.

Proof. Using the point-wise description of left Kan extensions, we have that the desired left Kan extension L' is given by

$$L'(e,(A,(d_a))) \simeq \underset{\substack{\mathcal{E}\times(\mathsf{F}\wr\mathcal{C}^{op})^{op}\downarrow(e,(A,(d_a))) \\ (e,(B,(x_b)))\leftarrow(e,(A,(d_a)))}}{\mathrm{colim}} Y(e) \otimes \bigotimes_{b\in B} X(x_b) \simeq Y(e) \otimes \underset{(\mathsf{F}\wr\mathcal{C}^{op})^{op}\downarrow(A,(d_a))}{\mathrm{colim}} \bigotimes_{b\in B} X(x_b).$$

However, every map $(f,(f_a)) : (A,(d_a)) \to (B,(x_b))$ factors through $(A,(x_{f(a)}))$, and hence $\prod_a(\mathcal{C}^{op} \downarrow d_a)$ is a reflexive subcategory of $(\mathsf{F} \wr \mathcal{C}^{op})^{op} \downarrow (A,(d_a)))$, and hence is final. Thus, continuing our equation, we have

$$\simeq Y(e) \otimes \underset{\Pi(\mathcal{C}^{op}\downarrow d_a)}{\mathrm{colim}} \bigotimes_{a\in A} X(c_a).$$

The result then follows by using the fact that $\otimes$ commutes with limits to compare this versions the explicit description of the given composite.

□

Many of our indexing categories of structured or labeled trees are built as pullbacks involving the $\mathsf{F} \wr (-)$ construction. In many settings, these pullbacks also act well with left Kan extensions.

Lemma A.1.9. *Consider the following pullbacks of categories:*

$$\begin{tikzcd}
\mathbb{T}_1 \arrow[r] \arrow[d] & \mathsf{F}_0 \wr \mathbb{T}_0 \arrow[d] \\
\mathbb{T}_0 \arrow[r] & \mathsf{F}_0 \wr \Sigma
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{T}_{G,1} \arrow[r] \arrow[d] & \mathsf{F} \wr \mathbb{T}_{G,0} \arrow[d] \\
\mathbb{T}_{G,0} \arrow[r] & \mathsf{F} \wr \Sigma_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\lambda_1^n \mathbb{T}_1 \arrow[r] \arrow[d] & (\mathsf{F}_0^{\wr}\Sigma)^{\times n} \times \mathsf{F}_0 \wr \mathbb{T}_1 \arrow[d] \\
\lambda_0^n \mathbb{T}_0 \arrow[r] & (\mathsf{F}_0 \wr \Sigma)^{\times n} \times \mathsf{F}_0 \wr \mathbb{T}_0
\end{tikzcd}$$

In each case, right Kan extensions are preserved by the pullback.

Proof. We show the result for the first diagram; the rest are completely analogous. Unpacking definitions, the pointwise formula for right Kan extensions yields that it suffices to check that for each $T \in \mathbb{T}_0$, the functor

$$T \downarrow \mathbb{T}_1 \to V(T) \downarrow \mathsf{F}_0 \wr \mathbb{T}_0$$

is initial. We first observe that $\prod_{v \in V(T)} T_v \downarrow \mathbb{T}_0$ is initial in $V(T) \downarrow \mathsf{F}_0 \wr \mathbb{T}_0$. Similarly, $T \downarrow_{\simeq} \mathbb{T}_1$ is initial in $T \downarrow \mathbb{T}_1$, where the former is the subcategory spanned by those arrows $T \to d_1(S)$ which are isomorphisms of trees.

Finally, we have a natural isomorphism $T \downarrow_{\simeq} \mathbb{T}_1 \simeq \prod\limits_{v \in V(T)} T_v \downarrow \mathbb{T}_0$. Indeed, the left hand side encodes replanarizations of T equipped with assmebly data, while the right hand side encodes replanarizations of the vertices of T equipped with assembly data. By Proposition 2.3.41, we're done. $\square$

Appendix B

Counterexample to N_∞ Realization Candidate

We come back to the discussion of N_∞-operads from Section 4.2.2. In particular, Blumberg-Hill have shown that the coefficient system associated to any N_∞-operad is an indexing system. Further work by Blumberg-Hill [BH16] has shown that indexing systems also algebraically capture "norm" information via incomplete Tambara functors. Thus, it is very natural to expect that, for any indexing system $\mathcal{F}$, there is some N_∞-operad $N^{\mathcal{F}}$ with associated coefficient system $\mathcal{F}$.

In their paper, Blumberg-Hill describe a candidate for a categorical model of $N^{\mathcal{F}}$, based on the categorical Barratt-Eccles operad of [GMM12], a categorical model for a complete N_∞-operad (i.e. a G-E_∞-operad). However, we will show that this model fails in two ways:

(1) to have the correct homotopy type, and

(2) to be a suboperad of the Barratt-Eccles operad.

B.1 Introduction

B.1.1 Barratt-Eccles operad

We begin by recalling the *categorical Barratt-Eccles operad*:

Definition B.1.1. Let $\mathrm{Ob}_* : \mathsf{Set} \to \mathsf{Cat}$ be the right adjoint of the "objects" forgetful functor. Explicitly, $\mathrm{Ob}_*(X)$ has object set X, and a unique arrow between any two objects.

These have also been called "chaotic categories" [GMM12].

Example B.1.2. If G is a group, then $\mathrm{Ob}_*(G)$ is equivalent to $B_G G$, the translation groupoid of G.

In fact, we have the following classic result.

Lemma B.1.3. *The realization $|\mathrm{Ob}_*(G)|$ of the chaotic category generated by a group is a model for EG.*

These can be assembled to form operads; in particular, this is a simple way to construct an E_∞-operad.

Definition B.1.4. The collection $\{\Sigma_n\}$ is an operad in sets; thus, $\{\mathrm{Ob}_*(\Sigma_n)\}$ is an operad is categories, called the *Barratt-Eccles operad.* The realization is a model for an E_∞-operad.

Generalizing to the equivariant setting, the above can be paralleled to great affect.

Definition B.1.5 ([GMM12])**.** The collection $\{\mathsf{Set}(G, \Sigma_n)\}$ forms an operad in G-sets by post-composition with the operad $\{\Sigma_n\}$. Define $\mathbb{O}_n = \mathrm{Ob}_*\mathsf{Set}(G, \Sigma_n)$, and call the G-categorical operad $\mathbb{O} = \{\mathbb{O}_n\}$ the *(categorical) equivariant Barratt-Eccles operad.*

Theorem B.1.6 ([GMM12])**.** *The realization of $\mathbb{O}_n$ is a universal (G, Σ_n)-bundle; equivalently, $\mathbb{O}_n$ is a universal space for the family of all graph subgroups of $G \times \Sigma_n$. That is, the realization of $\mathbb{O}$ is a G-E_∞-operad of spaces.*

B.1.2 The Candidate

Now, we would like to form universal spaces for smaller families of graph subgroups, retaining operadic structure. The following is a natural choice to investigate.

Definition B.1.7. For any indexing system $\mathcal{F} = \{\mathcal{F}_n\}$, define

$$\mathsf{Set}_{\mathcal{F}}(G, \Sigma_n) := \{f \in \mathsf{Set}(G, \Sigma_n) \mid \mathrm{Stab}(f) \in \mathcal{F}_n\},$$

and let $\mathbb{O}_n\mathcal{F}$ be the subcategory $\mathrm{Ob}_*\mathsf{Set}_{\mathcal{F}}(G, \Sigma_n) \subseteq \mathrm{Ob}_*\mathsf{Set}(G, \Sigma_n)$.

Conjecture B.1.8 ([BH15], The Model Conjecture). *The realization* $|\mathbb{O}_n^{\mathcal{F}}| \simeq E\mathcal{F}_n$.

Moreover:

Conjecture B.1.9 ([BH15], The Suboperad Conjecture). *If* $\mathcal{F} = \{\mathcal{F}_n\}$ *is an indexing system, then the categories* $\mathbb{O}_*^{\mathcal{F}}$ *form a suboperad of the Barratt-Eccles operad* $\mathbb{O}_*$.

A direct consequence of these two conjectures would be a (different) proof of Conjecture 4.2.22: sending C to the operad $\mathbb{O}^{\mathcal{F}}$ would be a clear inverse of the functor $\mathrm{Ho}(N_\infty\text{-}\mathsf{Op}) \to \mathbb{I}$.

However, the above results are false for general groups and indexing systems.

Proposition B.1.10. *Conjecture B.1.8 is false for generic groups* G*; e.g.* $G = \mathbb{Z}/2 \times \mathbb{Z}/2$.

Moreover, even if this conjecture holds for a particular group G, the resulting symmetric G-sequences may not be a suboperad:

Proposition B.1.11. *There exist groups* G *and indexing systems* $\mathcal{F} = \{\mathcal{F}_n\}$ *over* G *for which the following hold simultaneously:*

- $\mathbb{O}_n^{\mathcal{F}}$ *is a model for* $E\mathcal{F}_n$,
- $\mathbb{O}^{\mathcal{F}}$ *is not a suboperad of* $\mathbb{O}$.

B.1.3 Computing Stabilizers

Due to our definition of $\mathbb{O}^{\mathcal{F}}$, much of the discussion will revolve around calculations of stabilizers. We have:

Lemma B.1.12. *For any* $f \in Set(G, \Sigma_n)$*, we have*

$$\mathrm{Stab}(f) = \left\{(h, f(h)^{-1}f(1)) \mid f(hx) = f(x)f(1)^{-1}f(h) \textit{ for all } x \in G\right\}.$$

Proof. g Suppose $(g,\sigma) \in \mathrm{Stab}(f)$, so $f(x) = ((g,\sigma).f)(x) = f(g^{-1}x)\sigma^{-1}$ for all $x \in G$. In particular, taking $x = g$, we find $\sigma = f(g)^{-1}f(1)$, and then taking $x = gx$, we produce $f(gx) = f(x)f(1)^{-1}f(g)$. □

We write $H_f = \{h \in G \mid f(hx) = f(x)f(1)^{-1}f(h) \text{ for all } x \in G\}$; then $\mathrm{Stab}(f) = \Gamma(f|_{H_f})$.

Remark B.1.13. Two warnings:

(1) Even if $f(1) = 1$, this H_f is *not* necessarily the largest subgroup H of G such that $f|_H$ is a group anti-homomorphism; clearly $H_f \leq H$, but they do not have to be equal. Consider the example where $G = C_4 = \{1, t, t^2, t^3\}$, $k >> 0$ so that $\{z^n \mid 0 \leq n \leq 7\} = C_8 \leq \Sigma_k$, and we have a map $f : G \to \Sigma_k$ defined by

$$
\begin{aligned}
1 &\mapsto 1 \\
t &\mapsto z \\
t^2 &\mapsto z^4 \\
t^3 &\mapsto z.
\end{aligned}
$$

Then the largest H_f such that $f(hg) = f(g)f(h)$ for all $h \in H_f$, $g \in G$ is the trivial subgroup. Indeed, $f(t \cdot t) \neq f(t)f(t)$, $f(t^2 \cdot t) \neq f(t)f(t^2)$, and $f(t^3 \cdot t) \neq f(t)f(t^3)$. However, $f|_{\{1,t^2\}}$ is clearly a group (anti-)homomorphism.

(2) The stabilizer of $f|_K$ for some subgroup $K \leq G$ can again be larger than $H_f \cap K$; the previous example also shows this, with $K = \{1, t^2\}$.

We will also need to know when set maps f are fixed by subgroups Λ:

Lemma B.1.14. *For $\rho \in \mathrm{Hom}(H, \Sigma_n)$ and $f \in \mathsf{Set}(G, \Sigma_n)$, $\Gamma(\rho) \leq \mathrm{Stab}(f)$ if and only if $f(hx) = f(x)\rho(h)^{-1}$ for all $x \in G$ and $h \in H$.*

Proof. Assuming f is stabilized by $\Gamma(\rho)$, we have $\rho(h) = f(h)^{-1}f(1)$ for all $h \in H$. Thus, for all $h \in H$,

$$f(hx) = f(x)f(1)^{-1}f(h) = f(x)f(1)^{-1} \cdot f(1)\rho(h)^{-1} = f(x)\rho(h)^{-1}.$$

Conversely, we have $f(h \cdot 1) = f(1)\rho(h)^{-1}$, so $\rho(h)^{-1} = f(1)^{-1}f(h)$, and thus $f(hx) = f(x)\rho(h)^{-1} = f(x)f(1)^{-1}f(h)$, as desired. $\square$

Definition B.1.15. We call a set map f with $\Gamma(\rho) \leq \mathrm{Stab}(f)$ a *stabilizer extension of* ρ.

Lemma B.1.16. *If* $f \in \mathsf{Set}(G, \Sigma_n)$ *is a stabilizer extension of* $\rho \in \mathrm{Hom}(H, \Sigma_n)$*, then* $g_i \in H_f$ *if and only if* $Hg_i \subseteq H_f$*.*

Proof. We have $f(g_i x) = f(x)f(1)^{-1}f(g_i)$ for all $x \in G$, and moreover $f(hx) = f(x)\rho(h)^{-1}$ for all $x \in G$ and $h \in H$. Thus

$$f(hg_i x) = f(g_i x)\rho(h)^{-1} = f(x)f(1)^{-1}f(g_i)\rho(h)^{-1} = f(x)f(hg_i).$$

□

Lemma B.1.17. *If there exists* $f \in \mathsf{Set}(G, \Sigma_n)$ *with* $\mathrm{Stab}(f) = \Lambda$*, then there exists* $\tilde{f} \in \mathsf{Set}(G, \Sigma_n)$ *with* $\tilde{f}(1) = 1$ *and* $\mathrm{Stab}(\tilde{f}) = \mathrm{Stab}(f) = \Lambda$*.*

Proof. We let $\tilde{f}(x) = f(1)^{-1}f(x)$. The verification is straight-forward. □

Remark B.1.18. These last three lemmas imply that if we are trying to build a stabilizer extension of ρ, we only need to choose values for f on a transversal $\{g_i\}$ of $H\backslash G$ with $g_1 = 1$ and $f(1) = 1$; indeed, we then *must* define f by $f(kg_i) = f(g_i)\rho(k)^{-1}$. That is, f must "repeat" itself (shifted by the values of $f(g_i)$) on cosets of H.

B.2 The Model Conjecture

B.2.1 A Counterexample

For determining whether $|\mathbb{O}_n^{\mathcal{F}}|$ is a universal space for the family $\mathcal{F}_n$, it suffices to check that $\mathsf{Set}_{\mathcal{F}}(G, \Sigma_n)^{\Lambda} \neq \varnothing$ if and only if $\Lambda \in \mathcal{F}_n$. Indeed, since $\mathbb{O}_n^{\mathcal{F}}$ is a chaotic category, it is a connected groupoid where every element has trivial automorphism group; thus its realization is contractible if and only if it is non-empty. Moreover, since fixed points commute with geometric realization and right adjoints, we have

$$|\mathbb{O}_n^{\mathcal{F}}|^{\Lambda} = |i^*\mathsf{Set}_{\mathcal{F}}(G, \Sigma_n)|^{\Lambda} = |i^*(\mathsf{Set}_{\mathcal{F}}(G, \Sigma_n)^{\Lambda})|.$$

We now give a counterexample to Conjecture B.1.8, proving Proposition B.1.10:

Example B.2.1. Let $G = C_2 \times C_2$, and consider the indexing system defined by letting $C(1 \times 1) = C(1 \times C_2) = C(C_2 \times C_2)$ be just the trivial H-sets, and $C(C_2 \times 1) = \{\coprod_n C_2 \times 1\}_{n \in \mathbb{N}} \cup \mathbb{N}$. Then, in particular $\Lambda = \Gamma(\rho) = \{((1,1),1),((\tau,1),\tau)\}$ is in $\mathcal{F}_2$, where τ is the non-trivial element in $C_2 = \Sigma_2$ and ρ the obvious associated non-trivial homomorphism $C_2 \times 1 \to \Sigma_2$. However, any map $\tilde{\rho} \in Set(C_2 \times C_2, \Sigma_2)$ having $\Lambda = \Gamma(\rho) \leq Stab(\tilde{\rho})$ has a strictly *larger* stabilizer. Indeed, no matter where we send $(1,\tau)$ and (τ,τ), they will both be in the stabilizer:

$$\begin{aligned} (1,1) &\longmapsto 1 \\ (\tau,1) &\longmapsto \tau \\ (1,\tau) &\longmapsto x \\ (\tau,\tau) &\longmapsto \tau \cdot x; \end{aligned}$$

a	$\tilde{\rho}((1,\tau)\cdot a)$	$\tilde{\rho}(a)\cdot\tilde{\rho}((1,\tau))$
$(1,1)$	x	$1 \cdot x = x$
$(\tau,1)$	τx	$\tau \cdot x$
$(1,\tau)$	1	$x \cdot x = 1$
(τ,τ)	τ	$\tau x \cdot x = \tau$

for x either 1 or τ. Thus any map in $\mathsf{Set}(G,\Sigma_n)^\Lambda$ has stabilizer $\Gamma(\beta)$ for some $\beta \in Hom(G,\Sigma_2)$, and hence cannot be in $\mathcal{F}_2$. Thus $\mathsf{Set}_{\mathcal{F}}(G,\Sigma_n)^\Lambda = \varnothing$ for some $\Lambda \in \mathcal{F}_2$, and hence $|\mathbb{O}_2^{\mathcal{F}}|$ is not a universal space for $\mathcal{F}_2$.

B.2.2 Property (E) and the Model Conjecture

The above subsection showed that our original guess will not work for all groups G and all the sequences $\mathcal{F} = \{\mathcal{F}_n\}$ of families we want to consider. Let us try to salvage something from $\mathbb{O}_n^{\mathcal{F}}$. In particular, we would like to know when $|\mathbb{O}_n^{\mathcal{F}}|$ **is** in fact a universal space for $\mathcal{F}_n$. Our counterexample shows what can go wrong: the map $f \mapsto Stab(f)$ might not have a large enough target. This is the defining feature of groups that will work.

Definition B.2.2. We say a group G satisfies *Property (E)* if for all $\rho \in \mathrm{Hom}(H, \Sigma_n)$ non-trivial, there exists $f \in \mathsf{Set}(G, \Sigma_n)$ such that $\mathrm{Stab}(f) = \Gamma(\rho)$

Equivalently, G satisfies Property (E) if the image of the map

$$\mathsf{Set}(G, \Sigma_n) \to \{\Lambda \leq G \times \Sigma_n \mid \Lambda \cap \Sigma_n = 1\}$$

given by $f \mapsto Stab(f)$ contains the subset $\{\Lambda \leq G \times \Sigma_n \mid \Lambda \cap \Sigma_n = 1\} \backslash \{H \times \{1\} \leq G \times \Sigma_n\}$ for all $n \in \mathbb{N}$.

Proposition B.2.3. *G satisfies Property (E) if and only if Conjecture B.1.8 holds for G.*

Proof. In the "only if" direction, we just need to show that $Set_{\mathcal{F}}(G, \Sigma_n)^{\Lambda}$ is non-empty. But since $\Lambda \in \mathcal{F}_n$ there exists $f \in Set(G, \Sigma_n)$ with $\mathrm{Stab}(f) = \Lambda$, and hence $f \in \mathsf{Set}_{\mathcal{F}}(G, \Sigma_n)^{\Lambda}$. Conversely, suppose we have $\rho \in \mathrm{Hom}(H, \Sigma_n)$ non-trivial such that for all $f \in \mathsf{Set}(G, \Sigma_n)$ with $\mathrm{Stab}(f) \leq \Gamma(\rho)$ (that is, $f \in \mathsf{Set}(G, \Sigma_n)^{\Gamma(\rho)}$), $\mathrm{Stab}(f)$ is strictly larger than $\Gamma(\rho)$. Then we construct the family F "generated" by $\Gamma(\rho)$ by collecting all subgroups, all conjugates, and all conjugates of subgroups. In particular, H and it's conjugates are maximal elements in the lattice of F. But since all $f \in \mathsf{Set}(G, \Sigma_n)^{\Gamma(\rho)}$ have stabilizers in a strictly higher stratum of the lattice, $Set_{\mathcal{F}}(G, \Sigma_n)^{\Gamma(\rho)}$ is empty. □

Remark B.2.4. Example B.2.1 above exactly says that $C_2 \times C_2$ does not have Property (E). Moreover, a similar argument shows that $G \times C_2$, G abelian and non-trivial, never has Property (E).

So the question now becomes: when does a group satisfy Property (E)? We start by looking at some extension properties: for fixed $\rho \in \mathrm{Hom}(H, \Sigma_n)$, are there some properties of the pair (G, H) that can allow us to construct an $f \in \mathsf{Set}(G, \Sigma_n)$ with $\mathrm{Stab}(f) = \Gamma(\rho)$?

By Lemma B.1.18, we only need to define our new set map f on a transversal of $H \backslash G$: $f(kg_i) := f(g_i)\rho(k)^{-1}$, $k \in H$ and g_i in our transversal.

A Sufficient Condition

Suppose we have a given fixed $\rho \in \mathrm{Hom}(H, \Pi)$.

Lemma B.2.5. *Suppose there exists $\pi_0 \in \Pi$ such that $\pi_0^2 \neq 1$, and there exists $h_0 \in Z(G)$ such that $\rho(h_0) \neq 1$. Then there exists a map $f \in \mathsf{Set}(G, \Sigma_n)$ such that $\mathrm{Stab}(f) = \Gamma(\rho)$.*

Proof. We will build our function $f \in \mathsf{Set}(G, \Pi)$ coset by coset, by choosing our representatives and their images carefully by induction, again setting $f(kg_i) = f(g_i)\rho(k)^{-1}$ for $k \in H$ and $\{g_i\}$ our chosen transversal.

For each $g_i \neq 1$, we will show that $g_i \notin H_f$, and by the above lemmas that will be enough.

We start by letting $g_1 = 1$ and letting $f(1) = 1$. Now, by induction, suppose we have choose $g_1, \ldots, g_{n-1}$ such that $g_i \in H_f$ if and only if $i = 1$, and let $g_n \in G \setminus (\cup_{i=1}^{n-1} Hg_i)$ be arbitrary.

Case I $Hg_n^{-1} = Hg_n$.

Let h_n be defined by $g_n^{-1} = h_n g_n$.

Case IA $\rho(h_n) \neq 1$.

Then define $f(g_n) = 1$. We observe that $g_n \notin H_f$:

$$1 = f(g_n \cdot g_n^{-1}) \neq f(g_n^{-1})f(g_n) = \rho(h_2)^{-1} \cdot 1,$$

Case IB $\rho(h_n) = 1$.

Then define $f(g_n) = \pi_0$. We observe that $g_n \notin H_f$:

$$1 = f(g_n \cdot g_n^{-1}) \neq f(g_n^{-1})f(g_n) = 1 \cdot \pi_0.$$

Case II $Hg_2^{-1} \neq Hg_i$ for any $i \in \{1, \ldots, n-1\}$.

Then define g_{n+1} such that $g_n^{-1} = h_0 g_{n+1}$ (i.e. $g_{n+1} = h_0^{-1} g_n^{-1}$), and let $f(g_n) =$

$f(g_{n+1}) = 1$. We observe that neither g_n nor g_{n+1} are in H_f:

$$1 = f(g_n \cdot g_n^{-1}) \neq f(g_n^{-1})f(g_n) = f(h_0 g_{n+1})f(g_n) = 1 \cdot \rho(h_0)^{-1} \cdot 1;$$

$$1 = f(g_{n+1} \cdot g_{n+1}^{-1}) \neq f(g_{n+1}^{-1})f(g_{n+1}) = f(g_n h_0)f(g_{n+1}) = f(h_0 g_n)f(g_{n+1}) = 1 \cdot \rho(h_0)^{-1} \cdot 1.$$

Case III $Hg_n^{-1} = Hg_i$ for some $i \in \{1, \ldots, n-1\}$; say $g_n^{-1} = h_n g_i$.

Case IIIA $\rho(h_n) \neq 1$.

Define $f(g_n) = f(g_i)^{-1}$. We observe that $g_n \notin H_f$:

$f(g_n^{-1})f(g_n) = f(h_n g_i)f(g_i)^{-1} = f(g_i)\rho(h_2)f(g_i)^{-1}$, and this equals $1 = f(g_n g_n^{-1})$ if and only if $\rho(h_n) = 1$, a contradiction.

Case IIIB $\rho(h_n) = 1$.

Define $f(g_n)$ to be 1 if $f(g_i) = \pi_0$ or π_0^{-1}, and π_0 if $f(g_i) = 1$. We observe that $g_n \notin H_f$:

$$1 = f(g_n \cdot g_n^{-1}) \neq f(g_n^{-1})f(g_n) = f(h_n g_i)f(g_n) = f(g_i)f(g_n).$$

□

This again may seem restrictive, but for abelian groups it simplifies matters greatly:

Corollary B.2.6. *If G is abelian, we only need to check that Property (E) holds for the case $n = 2$.* □

Proposition B.2.7. *Cyclic groups satisfy Property (E).*

Proof. Let $G = C_n =< t >$, and $H \leq G$, say $H =< t^m >$, with m minimal, and let b be such that $n = mb$. Let $\rho \in \mathrm{Hom}(H, \Sigma_2)$.

Case I: $m = 1$.

Choose $f = inv \circ \rho$.

Case II: $m \neq 1$.

Since ρ is non-trivial, $\rho(t^m) = \pi \neq 1$ and $\rho((t^m)^k) = \pi^k$. We have a transversal $\{1, t, t^2, \ldots, t^{m-1}\}$ for G/H, and we define $f(t^l) = \pi$ if $l \neq 0$, and $f(t^0) = f(1) = 1$. Thus, globally, we have $f(t^{mk+l}) = \pi^\epsilon \pi^k$, for $0 \leq l \lneq m$, $0 \leq a \lneq b$, with $\epsilon = 0$ if $l = 0$ and $\epsilon = 1$ if $l \neq 0$. In particular, this satisfies $f(hx) = f(x)\rho(h)^{-1}$ for all $h \in H$, $x \in G$ (since every element of C_2 is its own inverse). By Lemma B.1.16, it suffices to check that these t^l are not in H_f unless $l = 0$. If t^l were in H_f, then in particular we would have $f(t^l \cdot t^{m-l}) = f(t^{m-1})f(t^l)$. Since $l \neq 0$, both $f(t^l)$ and $f(t^{m-l})$ are equal to π, and hence the right hand side is equal to $\pi^2 = 1$. However, the left hand side is $f(t^m) = \pi$. Hence no non-trivial t^l are in H_f, and hence H_f is precisely H, so $\mathrm{Stab}(f) = \Gamma(\rho)$, as desired.

□

B.3 Counterexample to the Suboperad Conjecture

We can now describe a family of counterexamples to Conjecture B.1.9 which proves Proposition B.1.11:

Example B.3.1. Let $G = C_{2N}$ be any even-ordered cyclic group. Consider the map $\varphi \in \mathsf{Set}(C_{2N}, \Sigma_3)$ which sends t^{2m} to 1, and t^{2m+1} to $\sigma := (1\ 3\ 2)$ for all $0 \leq m < N$. Then, a straightforward calculation shows that $\mathrm{Stab}(\varphi) =< t^2 > \times 1$, where $< x >$ is the subgroup generated by the element x. Now, let f_1, f_2, and f_3 be trivial maps from C_2 to Σ_5, Σ_3, and Σ_2, respectively, and $\gamma := \gamma(\varphi; f_1, f_2, f_3)$ be their Barratt-Eccles operadic composition. We compute that $\gamma(t^{2m})$ equals 1, and $\gamma(t^{2m+1})$ equals the block permutation $\tau := (1\ 3\ 2)(5, 3, 2)$. However, since $\tau^2 = 1$, H_γ is all of C_{2N}, and in particular is the graph of a non-trivial homomorphism out of G.

Thus, if C is any indexing system for $G = C_{2N}$ such that $C(G)$ contains only the trivial G-sets, then this lands outside $\mathsf{Set}_{\mathcal{F}}(C_{2N}, \Sigma_*)$, and hence $\mathsf{Set}_{\mathcal{F}}(C_{2N}, \Sigma_*)$ is not a suboperad of the Barratt-Eccles operad $\mathsf{Set}(C_{2N}, \Sigma_*)$.

Remark B.3.2. This example is fairly ad-hoc, and we can create many other similar families of counterexamples. Moreover, this counterexample came from composing set functions with *trivial* graphs as stabilizers. This just emphasizes the fact that the stabilizers exert very little control over the composition.

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